Overspill and forcing*

Jindřich Zapletal †
Academy of Sciences, Czech Republic
University of Florida

August 25, 2011

Abstract

The overspill phenomenon in descriptive set theory corresponds to a forcing preservation property, with a fusion type infinite game associated to it. As an application, it is consistent with the axioms of set theory that the circle $\mathbb{T}$ can be covered by $\aleph_1$ many closed sets of uniqueness while a much larger number of H-sets is necessary to cover it.

1 Introduction

Workers in mathematical logic occasionally use overspill type arguments: if $A \subset B$ are two sets of different true complexities, then the set $B \setminus A$ must be nonempty. This conclusion is particularly attractive in the case that a direct construction of elements in $B \setminus A$ is combinatorially involved. In this paper, I will be interested in the following version of overspill:

Definition 1.1. Let X be an uncountable compact metric space and let I be a hereditary collection of subsets of X. Say that I has the *overspill property* if there is no analytic set $A \subset K(X)$ such that $K_\omega(X) \subset A \subset I$.

Here, I is hereditary if it is closed under taking a subset. $K(X)$ is the hyperspace of all compact subsets of X equipped with the Vietoris topology, and $K_\omega(X) \subset K(X)$ is the collection of all countable compact subsets of X. Thus, if I has the overspill property and contains all compact countable sets, then every analytic collection of compact sets containing all countable compact sets must contain also a compact set which is not in I.

As the most trivial examples, the set $K_\omega(X)$ itself has the overspill property, since it is not analytic, and the collection of sets null with respect to some fixed Borel probability measure fails to have the overspill property, since it is G_δ in

*2000 AMS subject classification 03E40, 42A99.

†Partially supported by NSF grant DMS 0801114 and Institutional Research Plan No. AV0Z10190503 and grant IAA100190902 of GA AV ČR.

$K(X)$. As a much less trivial example, Pelant and Zelený [8] showed that the σ-ideal generated by porous sets has the overspill property. This yields a compact non-σ-porous subset of the unit circle which is a set of uniqueness, since the collection of sets of uniqueness fails to have the overspill property by an old result of Loomis [7].

I will prove that in the case that $I \cap K(X)$ is coanalytic, there is an infinite two player game characterizing the overspill property–Theorem 2.2. From the forcing point of view, in the common case that I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal such that the quotient poset P_I of Borel I-positive sets ordered by inclusion is proper, this game immediately evokes fusion type arguments for this poset. It turns out that there is a corresponding forcing preservation property that persists under countable support product of such quotient posets–Theorems 3.6, 3.7, and 3.8. The correspondence beween overspill and forcing also immediately yields many new examples of σ-ideals with overspill property.

As a sample application, I will compare the cardinal invariants of two σ-ideals from harmonic analysis: the σ-ideal H_σ generated by H-sets, and the σ-ideal U_σ generated by closed sets of uniqueness. Recall that the covering number of an ideal I, $\mathtt{cov}(I)$, is the smallest cardinality of a collection of sets in I whose union is the whole underlying space. Since $H_\sigma \subset U_\sigma$, it is clear that $\mathtt{cov}(U_\sigma) \leq \mathtt{cov}(H_\sigma)$. It turns out though that it is consistent with ZFC that $\mathtt{cov}(U_\sigma) = \aleph_1$ is much smaller than $\mathtt{cov}(H_\sigma)$–Theorem 5.3. The main point here is that despite the complicated combinatorial properties of the two σ-ideals, the proof of this independence result is in some sense canonical and uses hard results from harmonic analysis very efficiently. The poset needed is the countable support product of the quotient poset P_{H_σ}; the reason why the cardinal $\mathtt{cov}(U_\sigma)$ remains small in the extension is that H_σ has the overspill property, while U_σ fails it, and this distinction survives the various forcing manipulations necessary. In addition, the quotient poset P_{H_σ} is very similar to a poset isolated by Shelah in combinatorial form for a completely different purpose.

The paper is laid out as follows. In Section 2, I will formulate the two player infinite game characterization of the overspill property. Section 3 contains the translation of overspill into forcing properties of quotient posets, together with a product preservation theorem. In Section 4, I isolate and investigate the overspill ordinal, the countable ordinal that rates the complexity of the overspill property of a given collection I. Finally, Section 5, the sample application result on $\mathtt{cov}(U_\sigma) < \mathtt{cov}(H_\sigma)$ is proved.

The notation in the paper follows the set theoretic standard of [4]. As a canonical reference for harmonic analysis I use [5], for descriptive set theory [6], for definable forcing [12]. If I is a σ-ideal on a Polish space X then P_I denotes the partial order of Borel I-positive sets ordered by inclusion. For a Polish space X, $K(X)$ denotes its hyperspace, i.e. the space of compact subsets of X with the Vietoris topology. A subset $I \subset K(X)$ is hereditary if it is closed under taking subsets: $K \subset L \in I$ implies $K \in I$. A closure of a set A in a topological space is denoted by a bar: $\bar{A}$. A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every Polish space Y and every analytic set $D \subset Y \times X$ the set $\{y \in Y :$ the vertical section D_y of D above y is in the ideal $I\}$ is coanalytic.

2 The overspill game

Definition 2.1. Let X be a compact metric space with a fixed countable topology basis $\mathcal{O}$ closed under finite unions and intersections, and a fixed metric. Let $I \subset K(X)$ be a coanalytic collection of compact subsets of X, containing all countable compact sets and closed under subsets. The game $G(I)$ (or $G(I, X)$ if the underlying space is not clear from the context) is played between Player I and II for infinitely many rounds. In the n-th round of the game $G(I)$, Player I produces a countable compact set C_n and Player II responds with a basic open set $O_n \in \mathcal{O}$. The players must conform to the rules $K_n \subset O_n$, $K_{n+1} \subset O_n$, and $\bar{O}_{n+1} \subset O_n$. Player I wins if the *result of the play*, the intersection $\bigcap_n O_n = \bigcap_n \bar{O}_n$ does not belong to the collection I.

Clearly, I can and will assume that Player I is playing so that $K_0 \subset K_1 \subset \dots$, and Player II is playing so that every point of O_n is within 2^{-n}-distance of a point in K_n. That way, the result of the play is equal to the closure of the union $\bigcup_n K_n$.

Theorem 2.2. *Suppose that X is a compact metric space and I is a hereditary subset of $K(X)$. Then*

1. *I has the overspill property if and only if Player II has no winning strategy in the game $G(I)$;*
2. *if I is coanalytic then the game $G(I)$ is determined.*

Proof. To prove (1), suppose first that Player II has a winning strategy σ in the game $G(I)$. I must produce an analytic collection of compact subsets of X which is a subset of I and contains all countable compact subsets of X. By tree induction build a countable tree T of partial finite plays according to the strategy σ ending with a move of Player II, such that if $t \in T$ is a node and $O \in \mathcal{O}$ is a basic open set that strategy σ can produce in the next round after t is followed with some challenge of Player I, then there is an immediate successor $s \in T$ of the node t that indeed ends with the strategy σ playing the set O. Now, if $b \in [T]$ is a branch through the tree T, it is an infinite play against the strategy σ, so Player II won and the end result of it is in the collection I. Consider the set $A = \{C \in K(X) :$ for some branch $b \in [T]$, K is covered by the end result of the play $b\}$. This is an analytic collection of compact sets, and since I is closed under subsets, $A \subset I$. Moreover, all countable compact sets belong to A: if $C \in K(X)$ is countable, then by induction on n build nodes $t_n \in T$ so that C is a subset of the last move in the play t_n. The induction step is possible to perform, since C is a legal move of Player I in the next round past t_n and it must induce the strategy to answer with a set which is still a superset of C. The construction of tre tree T guarantees that there is an immediate successor t_{n+1} of t_n whose last move is still a superset of C as desired. In the end, the end result of the play $\bigcup_n t_n$ is a superset of C and shows that $C \in A$.

On the other hand, suppose that $A \subset K(X)$ is an analytic collection containing all countable compact sets, and $A \subset I$. I must produce a winning strategy σ for Player II. Let $g : \omega^\omega \to K(X)$ be a continuous function such that $A = \text{rng}(g)$. Player II will win by producing, along with the moves of the game, sequences $t_n \in \omega^n$ so that $0 = t_0 \subset t_1 \subset \dots$ and for every number $n \in \omega$, the following statement $(*n)$ holds: for every countable compact subset $K \subset O_n$ there is a point $y \in \omega^\omega$ extending t_n such that $K \subset g(y)$. That way, the end result $L \subset X$ of the play will be a subset of $g(y)$ where $y = \bigcup_n t_n$, and as $g(y) \in I$ and I is closed under subsets, $L \in I$ and Player II won. To see that $L \subset g(y)$, observe that L is the closure of $\bigcup_n K_n$; thus, if some point of L did not belong to the compact set $g(y)$, already some point of some K_n together with its whole open neigborhood would not belong to $g(y)$, and by the continuity of the function g there would have to be a number $m > n$ such that for every point $y' \in \omega^\omega$ with $t_m \subset y'$, the set $g(y')$ is disjoint from that open neighborhood, contradicting $(*m)$.

It is necessary to prove that Player II can maintain $(*)$ at every stage of the play. $(*0)$ by the assumptions on the set A no matter what the open set O_0 is. Now suppose that $(*n)$ holds and Player I produces a set K_{n+1}. I must show that there is an open set O containing K_n and a number $i \in \omega$ such that $(*n+1)$ holds with $O_{n+1} = O$ and $t_{n+1} = t_n^\frown i$. Suppose for contradiction that this is not the case. Choose inclusion decreasing basic open sets $\langle P_i : i \in \omega\rangle$ such that each of them is a legal move for Player II at this stage and $K_{n+1} = \bigcap_i P_i$. Since $(*n+1)$ must fail, there are countable compact sets $L_i \subset P_i$ such that for every $j \in i$ and every point $y \in \omega^\omega$ with $t_n^\frown j \subset y$ the inclusion $L_i \subset g(y)$ fails. Now, the closure L of the union $\bigcup_i L_i$ contains only the points in $\bigcup_i L_i$ and points in K_n, so in particular L is a countable compact subset of O_n. By the induction hypothesis, there must be a point $y \in \omega^\omega$ extending t_n such that $L \subset g(y)$. This, however, contradicts the choice of the set L_i where i is any number than the first entry of the sequence y past t_n!

(2) of the theorem is proved by a standard unraveling argument. Since the collection I is coanalytic, there is a continuous function $g : \omega^\omega \to K(X)$ whose image is the complement of I. Consider the game $G'(I)$ which is slightly more difficult than $G(I)$ for Player I. The game $G'(I)$ proceeds in the same way as the previous one, except in some rounds, Player I also indicates a natural number i_n. Player I wins he indicated infinitely many numbers, thereby creating a sequence $y \in \omega^\omega$, and $g(y) \subset \bigcap_n O_n$. Thus, if Player I wins in a play of the game $G'(I)$, then he also won the associated play of the game G: the set $g(y)$ is not in I, and as the collection I is closed under subsets, the set $\bigcap_n O_n$ cannot belong to I either. In the wide tree of all possible plays of the game $G'(I)$, the plays in which Player I wins forms a G_δ set, and the game $G'(I)$ is therefore determined. I will show that winning strategies for both players in the new game translate to winning strategies in the old game.

It is clear that if Player I has a winning strategy in the game $G'(I)$, then the same strategy, merely omitting the additional moves, will be his winning strategy in the game $G(I)$. Now suppose that σ' is a winning strategy for Player II in the game $G'(I)$. To get a winning strategy for this player in the original game, note that σ' can be easily improved not to depend on the choices of the numbers i_n as long as these numbers are smaller than the index of the round at which thay are played. Simply at each round consider the finitely many possibilities for such choices of these numbers in the previous round and play the intersection of all sets that σ' advises to play against each. I claim that this improved strategy σ is in fact winning for Player II in the original game $G(I)$. Indeed, if there is a play p in the game $G(I)$ against this strategy in which Player II loses, then the result L of that play cannot be in I and there is a point $y \in \omega^\omega$ such that $g(y) = L$. Consider the play p' against the strategy σ' in which Player I plays the same compact sets as in p and produces the point y in such a way that each number on it is added at a round with index larger than that number. The definition of the strategy σ implies that the moves of the strategy σ' in p' will be supersets of the corresponding moves of the strategy σ in p, therefore the moves of Player I in p' are legal and the result L' of the Play p' will be a superset of $L = g(y)$, resulting in Player I's victory. This of course contradicts the choice of the strategy σ'. $\square$

As the most trivial example for Player I, he has a winning strategy if I is the collection of countable compact subsets of the Cantor space $X = 2^\omega$. He will win by playing finite sets C_n such that $C_0 \subset C_1 \subset \dots$ such that for every number n and every point $x \in C_n$ there is another point $x \neq y \in C_{n+1}$ such that x, y agree on the first n positions. In the end, the result of the play must contain the closure of the set $\bigcup_n C_n$, which is perfect, therefore uncountable and winning for Player I. Note the similarity between this winning strategy and the fusion arguments for the Sacks forcing (which is isomorphic to a dense subset of P_I).

As the most trivial example for Player II, he has a winning strategy if I =the Lebesgue null sets. He will simply make use of the fact that every countable set is null and at the n-th move, he will cover the move K_n with an open set of mass $\leq 2^{-n}$. In this way, the result of the play will be Lebesgue null and therefore winning for Player II.

As one simple corollary of the theorem, note that the overspill property is closed under unions of finitely many coanalytic hereditary sets: finitely many winning strategies for Player I can be combined by just taking unions of moves in each.

3 The forcing connection

The winning strategies for Player I in the overspill game certainly remind the alert reader of various forcing fusion arguments. To exploit this parallel, I will need to consider a natural hereditary version of the overspill property. All examples of overspill property in this paper in fact have the hereditary version as well, even though in some circumstances such as the σ-ideal of σ-porous sets

[8], this requires a nontrivial modification of the overspill proof.

Definition 3.1. A σ-ideal I on a Polish space X has the *hereditary overspill property* if for every compact set $K \subset X$, $K \notin I$, the σ-ideal $I \restriction K$ has the overspill property.

Theorem 3.2. *Let I be a σ-ideal on a Polish space X such that the quotient forcing P_I is proper and bounding. The following are equivalent:*

1. *I has the hereditary overspill property;*
2. *for every Polish space Y and every analytic set $A \subset K(Y)$ containing all countable compact sets, P_I forces Y to be covered by the ground model elements of the set A.*

When talking about σ-ideals with hereditary overspill property, I will always restrict the attention to σ-ideals such that the quotient forcing P_I is bounding. This natural restriction has an important side effect. While in general, the status of the hereditary overspill property may depend on the choice of the topology on the underlying space X, in the case that P_I is bounding this choice is immaterial as long as the topologies yields the same Borel structure. This follows from the fact that for every Borel I-positive set $B \subset X$ and any two such Polish topologies t_0, t_1, there will be an I-positive compact set $C \subset B$ on which t_1 and t_2 coincide [12, Theorem 3.3.2].

Proof. On one hand, if I does not have the hereditary overspill property, then there is a compact I-positive set $C \subset X$ such that I on C does not have the overspill property, and therefore there is an analytic collection $A \subset K(C)$ containing all countable compact subsets of C, all of whose elements are in the σ-ideal I. Clearly, $C \Vdash \dot{x}_{gen}$ does not belong to any ground model coded I-small sets and so in particular to any ground model coded elements of A, and (2) fails.

On the other hand, if (1) holds, Y, A are as in (2), and $B \in P_I$ is a condition and $\dot{y}$ is a P_I-name for an element of the space Y, I must find an element of A such that a condition stronger than B forced $\dot{y}$ to this element of A. By the bounding property of the quotient P_I, strengthening B if necessary I may assume that B is compact and that there is a continuous function $f : B \to Y$ such that $B \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$ [12, Theorem 3.3.2]. Use the hereditary overspill property to thin out B further if necessary to make sure that the ideal I restricted to B has the overspill property. Let $A' = \{C \subset B$ compact: there is $K \in A$ such that $f''C \subset K\}$. This is an analytic collection of compact subsets of C which contains all countable compact sets, since an image of countable set is countable and A contains all countable compact subsets of Y. The overspill property yields an I-positive set $C \subset K$ with $C \in A'$. There is $K \in A$ such that $f''C \subset K$, and $C \Vdash \dot{y} \in \dot{K}$ as required. ☐

Corollary 3.3. *If I is a σ-ideal on a Polish space with the hereditary overspill property such that the quotient forcing P_I is proper and bounding. Then P_I preserves Baire category.*

In other words, P_I forces that the set of ground model elements of the unit interval is still non-meager. Restated without the forcing relation, there is no I-positive compact set K and a Borel set $D \subset K \times [0,1]$ such that the horizontal sections of D are in the σ-ideal I while the vertical sections of the complement are meager. In the particular case under investigation in this paper, the σ-ideal H_σ, this follows already from the fact that it is σ-generated by closed sets.

Proof. By [1, Theorem 2.2.4], Baire category preservation is equivalent to P_I not adding an eventually different real. Since P_I is assumed to be bounding, this is equivalent to not adding a bounded eventually different real. So let $f \in \omega^\omega$ be a function and let Y be the space $\Pi_n f(n)$. Let A be the set of those compact sets $L \subset Y$ for which there is a function $g \in \omega^\omega$ such that g infinitely often meets every element of L. If I show that L is an analytic subset of $K(Y)$ and contains every countable compact subset of Y, the theorem will apply to show that P_I forces every function dominated by f to belong to a ground model coded set in A, and therefore infinitely many times equal to a ground model function as desired.

Now, it is clear that L contains every countable compact set, since an obvious diagonalization argument yields a function infinitely many times equal to every element of the compact set. The analyticity of A is slightly more challenging. I will show that for a compact set $L \in K(Y)$, the following are equivalent:

- $L \in A$;
- for every $n \in \omega$ there is a larger $m \in \omega$ and a function $g : [n, m] \to \omega$ such that every element of L has nonempty intersection with g.

Clearly, the second item yields an analytic, in fact G_δ, description of the set A. The second item also easily implies the first, since the finite functions it provides can be pieced together to give a function in ω^ω to which every element of L is inifinitely many times equal. On the other hand, if the second item fails for some $n \in \omega$ and $g \in \omega^\omega$ is a function, it is not difficult to show that there must be an element $h \in L$ such that g and h agree only at some entries below n, and therefore $L \notin A$. To find h, for every $m \in \omega$ larger than m use the failure of the second item to find a function h_m which disagrees with g on all values between n and m, and use the compactness of the set L to find an accumulation point $h \in L$ of the set $\{h_m : m \in \omega\}$. The function h has the required properties. □

Corollary 3.4. *If I is a σ-ideal on a Polish space with the hereditary overspill property such that the quotient forcing P_I is proper and bounding, then P_I does not add a random real.*

Proof. Of course, this follows from the previous corollary since the random forcing does not preserve the Baire category. Still, it is curious to see the precision of the complexity arguments at work. Let Y be the unit interval equipped with the Lebesgue measure λ. The set $A = \{C \in K(Y) : \lambda(C) = 0\}$ is analytic, in fact G_δ, and it contains all countable compact sets. Thus, in the P_I extension, the ground model coded elements of the set A still cover the unit interval, so every real in the unit interval belongs to a ground model coded compact sets and therefore cannot be random. $\square$

Corollary 3.5. *If I is a σ-ideal on a Polish space with the hereditary overspill property such that the quotient forcing P_I is proper and bounding, then in the P_I-extension, the circle $\mathbb{T}$ is covered by ground model coded closed sets of uniqueness.*

Proof. This is the main point of this paper. The collection of closed sets of uniqueness is coanalytic in $K(\mathbb{T})$, so the theorem cannot be applied directly to it. However, it has a suitable analytic, in fact $G_{\delta\sigma}$, subcollection U' containing all countable compact sets. The subcollection is defined for example in [5, Section IV.2, Proposition 8]; the fact that every countable compact sets belong to it was proved by Loomis [5, Section V.5, Theorem 5], [7]. Thus, in fact, P_I forces that the circle is covered by ground model elements of this analytic collection. $\square$

With a reformulation of overspill as a forcing preservation property, a question immediately arises whether it persists under the usual forcing operations. The game characterization of overspill leads to preservation theorems for the countable support product of definable forcings. There is a similar preservation theorem for the countable support iteration with essentially identical proof; as it is not needed in this paper, I omit it.

Theorem 3.6. *Let $I_n : n \in \omega$ be $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on respective compact metric spaces $X_n : n \in \omega$, such that the quotient forcings are proper and bounding. If each of the ideals I_n has the hereditary overspill property, then so does their product ideal $\Pi_n I_n$.*

Theorem 3.7. *Let κ be a cardinal and $\{I_\alpha : \alpha \in \kappa\}$ be $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on Polish spaces with the hereditary overspill property such that the quotient P_{I_α} are proper and bounding for every $\alpha \in \kappa$, and let P be the countable support product of the posets $\{P_{I_\alpha} : \alpha \in \kappa\}$. Then for every Polish space Y and every analytic set $A \subset K(Y)$ containing all countable compact sets, P forces Y to be covered by the ground model elements of the set A.*

This theorem has a minor strengthening expressed in terms of a suitable cardinal invariant. Define the *overspill number* $\mathfrak{os}$ as the supremum of the cardinal numbers $\min\{|B| : B \subset A, \bigcup B = X\}$ as X ranges over all Polish spaces and A ranges over all analytic subsets of $K(X)$ containing all countable compact sets. It is immediate that $\mathfrak{os}$ is not smaller than the dominating number–just choose $X = \omega^\omega$ and $A = K(X)$. It is also true that $\mathfrak{os}$ is not smaller than the uniformity of the meager ideal using the analytic family from Corollary 3.3. Since, as proved above, the Sacks property implies overspill, which in turn implies the preservation of covering by analytic families containing all countable sets, it seems plausible that $\mathfrak{os}$ is not greater than the cofinality of the null ideal, but I do not have a proof of that statement. The preservation theorem 3.7 can be improved to state the following:

Theorem 3.8. *Suppose that the Generalized Continuum Hypothesis holds, κ is a cardinal, and $I_\alpha : \alpha \in \kappa$ are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on Polish spaces with the hereditary overspill property such that the quotient forcings are proper and bounding. Then the countable support product of the posets $\{P_{I_\alpha} : \alpha \in \kappa\}$ forces $\mathfrak{os} = \aleph_1$.*

Proof. For the proof of Theorem 3.6, it is first necessary to make sense of the product ideal $I = \Pi_n I_n$. This is the ideal on $\Pi_n X_n$ generated by those Borel sets which do not contain a box of the form $\Pi_n B_n$ where $B_n \subset X_n$ is a Borel I_n-positive set for every number n. Since the posets entering the product are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, proper, bounding, and Baire category preserving by Corollary 3.3, this is indeed a σ-ideal by [12, Theorem 5.2.6], it is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the full support product is proper, bounding, and preserves Baire category, and it is naturally isomorphic to a dense subset of the quotient poset P_I.

I will start with the product of two ideals I_0 and I_1. Let I be the product ideal on the space $X_0 \times X_1$, and let $B \subset X_0 \times X_1$ be a Borel I-positive set. Thinning out if necessary, I may assume that in fact $B = C_0 \times C_1$ for some compact sets $C_0 \subset X_0, C_1 \subset X_1$ which are positive in the respective ideals and where Player I has a winning strategy σ_0, σ_1 in the respective overspill games by Theorem 2.2. I will find a winning strategy σ for Player I in the game $G(I, B)$. To specify the game completely, fix the basis on the product space which is in a nactural sense product of the bases on the spaces entering the product, so every set in the product basis is a finite union of products of sets in the bases on the spaces X_0, X_1 respectively. The n-th move K_n of the strategy σ is simply the product of the moves $K^0_n \times K^1_n$ that the strategies σ_0, σ_1 would produce in related plays of their respective games. Note that K_n, as a product of countable sets, is again countable and if O_n is a basic open set covering K_n, then there are basic open sets O^0_n, O^1_n on X_0, X_1 respectively such that $K \subset O^0_n \times O^1_n \subset O_n$. Thus, I may assume that Player II in fact plays such product sets and apply the strategies σ_0, σ_1 to the moves O^0_n, O^1_n to get the sets K^0_{n+1}, K^1_{n+1}. In the end, the result of the play on the product is a product of the results of the plays on each of the two coordinates, and so positive in the product ideal. This confirms that Player I always wins if he sticks to the product strategy.

The easy diagonalization argument necessary for the case of the product of infinitely many ideals is left to the reader.

□

Proof. The proof of Theorem 3.7 is just a routine massaging of the previous argument. Consider the κ product of posets $\{P_{I_\alpha} : \alpha \in \kappa\}$. Suppose for contradiction that Y is a Polish space and $A \subset K(Y)$ is an analytic set containing all countable compact sets. Let $\dot{y}$ be a P-name for an element of Y and $p \in P$ is a condition; I must find a set $C \in A$ and a condition $q \leq p$ which forces $\dot{y} \in \dot{C}$. Let M be a countable elementary submodel of a large enough structure containing $p, \dot{x}$. A standard argument shows that there are I_α-positive compact sets $\{K_\alpha : \alpha \in \kappa \cap M\}$ such that the product $L = \Pi_{\alpha\in\kappa\cap M} K_\alpha$ consists of M-generic sequences only for the product forcing meeting the condition p, and the function $g : L \to Y$ given by $g(\vec{x}) = \dot{y}/\vec{x}$ is continuous. Let $A' \subset K(L)$ be the collection of all compact subsets of L whose images are covered by sets in A; this is an analytic collection of sets containing all countable sets. By the hereditary overspill property of the product ideal $\Pi_{\alpha\in\kappa\cap M} I_\alpha$, there are compact sets $\{K'_\alpha : \alpha \in \kappa \cap M\}$ such that $K'_\alpha \subset K_\alpha$ and $L' = \Pi_{\alpha\in\kappa\cap M} K'_\alpha \in A'$. The g-image of L' is then covered by some set $C \in A$, and a review of the definitions shows that L' is a condition below p that forces $\dot{y} \in \dot{C}$ as desired. □

Proof. The additional degree of difficulty in Theorem 3.8 compared to Theorem 3.7 lies in the possibility that new Polish spaces and new analytic collections of compact sets are added by the product, and in theory they could send the cardinal $\mathfrak{os}$ up. I will prove the key claim that rules out this possibility, and leave the further routine details to the reader. The claim is used to prove the overspill property of the product poset $(\Pi_{\alpha\in a} P_{I_\alpha})^V$ in the model $V[G \cap \Pi_{\alpha\in b} P_{I_\alpha}]$, where $a, b \subset \kappa$ are disjoint countable sets.

Claim 3.9. *Suppose that I is a coanalytic collection of compact subsets of a compact metric space X, closed under subsets, with the overspill property. Suppose that $V[G]$ is a bounding extension of V. Then in $V[G]$, I^* has the overspill property, where I^* is the collection of those compact sets which do not have a ground model coded compact subset which is not in I.*

Note that in $V[G]$, I^* is typically properly larger than I, so the statement of the claim is nontrivial. To prove the claim, suppose that I^* does not have the overspill property in $V[G]$, so Player II has a winning strategy $\sigma^* \in V[G]$ in the game $G(I^*)$ by Theorem 2.2. By the same theorem, Player I also has a winning strategy $\sigma \in V$ in the game $G(I)$ in the ground model. Let $a \in V$ be the countable set of all finite sequences of countable compact sets that the strategy σ can possibly produce against some counterplays by Player II. As $V[G]$ is a bounding extension of V, there is a ground model function $h : a \to \mathcal{P}(\mathcal{O})$ that assigns to every sequence $t \in a$ a finite set $h(t)$ such that the move the strategy σ^* dictates, Player I having played t, is in this finite set. Now, in the ground model V consider the play p of the game $G(I)$ in which Player I observes the strategy σ and at each intermediate stage t, Player II plays a set which is a subset of the intersection of all those elements of $h(t)$ which are basic open subsets of X covering the last move of Player I. In the extension, consider the play p^* in which Player II follows the strategy σ^* and Player I plays the same sequence of compact sets as in p. The choice of the function h implies that p^* is indeed a legal play against the strategy σ^*. The resulting set in both of these plays is the closure of the union of the sets that Player I played. It is certainly coded in the ground model, since P is in the ground model, and it is not in I, since σ was a winning strategy for Player I in the ground model. Thus, Player II lost the play p^*, contradicting the assumption that the strategy σ^* was winning. □

The attentive reader should not fail to notice how overspill fits into the doctrine of [12, Section 3.10]. Many forcing properties can be restated as the bad player not having a winning strategy in a certain game. If the forcing in question is suitably definable, then the game in question is determined, and the winning strategies for the good player can serve as a tool for proving preservation theorems for product or iteration.

4 The overspill ordinal

If I is a coanalytic hereditary subset of $K(X)$ for a compact X, then Player I has a winning strategy in the overspill game. Since Player II has only countably many moves at his disposal, the Cantor-Bendixson ranks of the moves of Player I will be uniformly bounded by some countable ordinal. In the case of hereditary overspill property, this countable ordinal may depend on the choice of the compact I-positive subset of X, and the supremum of these ordinals may be ω_1; in the natural examples though, the overspill ordinal is the same for all I-positive compact subsets of X. In this section I investigate the influence this *overspill ordinal* on the forcing properties of the poset P_I, and I show that it can be arbitrarily large.

It is not difficult to find upper bounds on the overspill ordinal of the σ-ideals considered elsewhere in this paper. The σ-ideal of countable sets has overspill ordinal equal to 1–Player I has a winning strategy using just finite sets. The H_σ ideal has overspill ordinal equal to 2–Player II plays just finite unions of converging sequences with their limits. The product of the H_σ ideal of dimension n has the overspill ordinal equal to $1+n$. Note that the winning strategy for the product ideal calls for the product of sets sets played at each coordinate, which increases the rank the moves by n. With a little bit of effort, one can show that the number $1+n$ is exact. The countable support product of infinitely many copies of the H_σ ideal has overspill ordinal ω. Finally, the proof of [8] yields an upper bound of the overspill ordinal of the σ-porous ideal to be ω.

Theorem 4.1. *For every countable ordinal α there is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I_α on 2^ω generated by closed sets which has the hereditary overspill property, while the overspill ordinal is at least α.*

Proof. For every countable ordinal $\alpha \in \omega_1$, choose a map $f_\alpha : \omega \to \alpha$ such that f_α-preimage of any fixed ordinal in α is infinite. Call a set $a \subset \omega$ *α-small* if $f_\alpha \restriction a$ is a decreasing function; in particular, such sets must be finite. Let I_α be the σ ideal on 2^ω be the σ-ideal σ-generated by all sets of the form $A_{\vec{u}}$ where $\vec{u}$ is a countable sequence of finite functions such that $\mathrm{dom}(\vec{u}(n))$ is α-small set for every $n \in \omega$, $\max \mathrm{dom}(\vec{u}(n)) < \min \mathrm{dom}(\vec{u}(n+1))$, and $\mathrm{rng}(\vec{u}(n)) \subset 2$. The set $A_{\vec{u}}$ is defined by $\{x \in 2^\omega : \forall n \in \omega\ \vec{u}(n) \cap x \neq 0\}$. These will be the σ-ideals we will use. In order to prove the theorem, I will show that I_α is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal such that the poset P_{I_α} is bounding and it has the overspill property, and its overspill ordinal is close to α. This follows from a series of claims that are of independent interest.

Claim 4.2. *Every analytic I_α-positive set has a compact I_α-positive subset.*

Proof. Let $A \subset 2^\omega$ be an I-positive analytic set, $A = \text{rng}(g)$ for some continuous function $g : \omega^\omega \to 2^\omega$. Let $T \subset \omega^{<\omega}$ be a tree such that $g''[T] \notin I$ and also for every node $t \in T$, $g''[t \restriction t] \notin I$. I must produce a finitely branching tree $S \subset T$ such that $f''[S] \notin I$–the set $g''[S]$ is a continuous image of a compact set and therefore compact, and it is the sought compact I-positive subset of the analytic set A.

For every node $t \in T$, argue that there exists a natural number $n \in \omega$ such that for every α-small set $a \subset \omega$ such that $f_\alpha(\min(a)) \in \beta$ and every function $u : a \to 2$, if there is a path $z \in [T \restriction t]$ such that $g(z) \restriction a = u$ then there is such a path z with $z(|t|) \in n$. If this was not the case then for every number n there would be the offending α-small set a_n and a function $u_n : a_n \to 2$. A wellfoundedness argument shows that there is a finite set $b \subset \omega$ such that b is an initial segment of infinitely many of the sets a_n, and b is maximal such in the initial segment ordering. Then, there must be an infinite set $c \subset \omega$ and a function $v : b \to 2$ such that for every $n \in c$, b is an initial segment of a_n, $u_n \restriction b = v$, and $\max(a_n) \in \min(a_n \setminus b)$. By the definitions, there is a path $y \in [T \restriction t]$ such that $u_n \cap g(y) = 0$, where $n = \min(c)$; in particular, $v \cap f(y) = 0$. By the continuity of the function g, there is a node s properly extending t, an initial segment of y, such that for all paths $z \in [T \restriction s]$ it is the case that $v \cap g(z) = 0$. Since the set $g''[T \restriction s]$ is I_α-positive, for all but infinitely many $n \in c$ it must be the case that there is a path $z_n \in [T \restriction s]$ such that $u_n \cap g(z_n) = 0$. Any such a number $n \in c$ with $s(|t|) \in n$ together with the path z_n yields a contradiction with the choice of the set a_n and the function u_n.

Using the previous paragraph repeatedly, find a *guiding sequence* of numbers $\langle n_i : i \in \omega \rangle$ such that for every $i \in \omega$ and every node $t \in T$ of length i dominated by the guiding sequence, the number n_i works as in the previous paragraph. I claim that the tree $S \subset T$ of all nodes dominated by the guiding sequence has the requested property: $g''[S] \notin I$.

To do this, for every node $s \in S$ and every collection $\{a_n : n \in \omega\}$ of α-small sets with $\max(a_n) \in \min(a_{n+1})$, and $\{u_n : a_n \to 2\}$ of functions, I must find an extension $s' \supset s$ in S and a number $n \in \omega$ such that for every $z \in [T]$ with $s' \subset z$ it is the case that $u_n \cap g(z) = 0$. Once this is done, an easy induction argument will produce path $z \in [S]$ such that $g(z)$ avoids any given countable collection of generating sets of the σ-ideal I_α. To find the extension $s' \supset s$, note that the set $g''[T \restriction s]$ is I_α-positive, and so there must be a number $n \in \omega$ and a path $z_0 \in [T \restriction s]$ such that $u_n \cap g(z_0) = 0$. By induction on $i \in \omega$ build an increasing sequence of nodes $t_i \in T$ extending s of respective length $|s| + i$, dominated by the guiding sequence, and paths $z_i \in [T \restriction t_i]$ such that $u_n \cap g(z_i) = 0$. Let $z = \bigcup_i t_i$; it is clear that $z = \lim_i z_i$ is dominated by the guiding sequence, and $u_n \cap g(z) = 0$ by the continuity of the function g. By the continuity again, there must be an initial segment $s' \subset z$ such that for every $z' \in [T \restriction s']$ it is the case that $g(z') \cap u_n = 0$. This completes the proof of the claim. □

Claim 4.3. *I_α is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal.*

Proof. To begin, consider a compact I_α-positive set $K \subset 2^\omega$. If necessary, thin it out to a smaller I_α-positive compact set so that its intersections with open sets are either empty or I-positive. In such a set, all generating sets of the σ-ideal are relatively meager, and therefore every relatively nonmeager set is I_α-positive. Moreover, the statement that K is I_α-positive is just equivalent to the following formula (*K): K is nonempty and if $O \subset 2^\omega$ is a basic open set, then either $K \cap O$ is empty, or there is an $n \in \omega$ such that for every function $u : a \to 2$ whose domain is an α-small set of natural numbers larger than n there is $x \in K \cap O$ such that $u \cap x = 0$.

Towards the proof of the claim, let $A \subset 2^\omega \times 2^\omega$ be an analytic set; I must show that the set $\{y \in 2^\omega : A_y \notin I_\alpha\}$ is analytic. Fix a closed set $D \subset 2^\omega \times 2^\omega \times \omega^\omega$ projecting into A. I will show that $A_y \notin I_\alpha$ if there is a compact set $K \subset X$ with (*K) and a continuous function $g : K \to D$ such that the first coordinate of $g(x)$ is y and the second x. This is an analytic statement.

Certainly the existence of K, g implies $A_y \notin I_\alpha$, since the function g provides the necessary witnesses for $K \subset A_y$ and (*K) guarantees the positivity of K. For the opposite implication, assume that $A_y \notin I_\alpha$. In order to find the compact set K and the continuous function g, use the previous claim to find a compact I_α-positive subset $K_0 \subset A_y$. Thin it out if necessary so that its intersections with open sets are either empty or I_α-positive. Use Jankov-von Neumann uniformization theorem [6, Theorem 18.1] to find a Baire-measurable function $g : K_0 \to D$ providing witnesses for $K_0 \subset A_y$. The function is continuous on a comeager subset of K_0, which is still I_α-positive, and contains a compact I_α-positive set $K \subset K_0$ with (*K). The pair K, g is as required.

□

Claim 4.4. *The σ-ideal I_α has the hereditary overspill property, and Player I has a winning strategy in the overspill game that uses only sets of Cantor-Bendixson rank $\leq \alpha$.*

Proof. First, by induction on $\beta \leq \alpha$, for all compact I-positive sets $K \subset 2^\omega$ simultaneously, prove that there is a countable compact set $L \subset K$ of rank at most β such that for every α-small set $a \subset \omega$ such that $f_\alpha(\min(a)) \in \beta$ and every $u : a \to 2$, if there is a point $x \in K$ such that $x \cap u = 0$ then there is such a point already in L.

The induction is straightforward. To deal with the limit stage of the induction, let γ be limit and assume that for all ordinals $\beta \in \gamma$, the statement holds. Fix the compact I-positive set K and thin it out if necessary so that every open neighborhood has either empty or I_α-positive intersection with K. Let $y \in K$ be an arbitrary point. Find numbers $n_i : i \in \omega$ such that for every $i \in \omega$ and every α-small set $a \subset \omega$ with $\min(a) > n_i$ and every function $u : a \to 2$, there is a point $x \in K$ such that $x \restriction i = y \restriction i$ and $x \cap u = 0$. Such numbers exist by the definition of the σ-ideal I_α. Now, for every binary sequence $t \in 2^\omega$ of length n_i such that $t \restriction i = y \restriction i$ and $K \cap [t] \neq 0$ use the induction hypothesis to find a countable compact set $K_t \subset K \cap [t]$ of Cantor-Bendixson rank $< \gamma$ such that for every α-small set $a \subset \omega$ such that $\min(a) \in n_i$ and $f_\alpha(\min(a)) \in \gamma$ and every function $u : a \to 2$, if there is a point $x \in K \cap [t]$ such that $x \cap u = 0$ then there is such a point in L_t. It is not difficult to show that the compact sets L_t tend to the point y, and the set $\bigcup_t L_t \cup \{y\}$ is the desired compact subset of K. The handling of the successor stage is very similar.

Now for the description of the winning strategy in the overspill game for Player I. Let $K \subset 2^\omega$ be a compact I_α-positive set, thinning it out if necessary we may assume that it has either empty or I_α-positive intersections with every open set. As the play progresses, Player I plays sets $K_n \subset K : n \in \omega$ and maintains lists L_n so that L_n is a finite set of finite binary sequences of length at least n such that the answer O_n of Player II at round n is equal to $\bigcup_{t \in L_n}[t]$. The next move K_{n+1} will then be a subset of $O_n \cap K$ of Cantor-Bendixson rank at most α such that whenever $t \in L_n$, $a \subset \omega$ is α-small, and $u : a \to 2$ is a function such that if there is a point $x \in K$ with $t \subset x$ and $u \cap x = 0$, then there is such a point in K_{n+1}. Player I can do this by the work of the previous paragraph, and it is not difficult to show that this strategy is winning for him. □

The most demanding part of the proof is showing that there are no winning strategies for Player I that use only moves of small rank. This is reflected in a certain forcing preservation property:

Claim 4.5. *Suppose that J is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space Y such that the quotient forcing P_J is proper and bounding, and moreover, the hereditary overspill property holds with Player I having a winning strategy in the overspill game using only moves of Cantor-Bendixson rank smaller than α. Then P_J forces that 2^ω is covered by the ground model coded sets in the σ-ideal $I_{\omega\alpha}$.*

Proof. Suppose that $B \in P_J$ is a condition and $\dot{y}$ is a P_J-name for an element of 2^ω. Use the bounding assumption to find an I-positive compact set $C \subset B$ and a continuous function $h : C \to 2^\omega$ such that $C \Vdash \dot{y}$ =the image of the generic point under h. Consider a play against the winning strategy of Player I in the overspill game in which Player I produces countable compact sets $K_n \subset C$ of Cantor-Bendixson rank $< \alpha$, and Player II chooses open sets O_n using the following procedure.

To obtain O_n, fix the number n. By induction on $i \in \omega$ find ordinals β_i, numbers $m_i \in \omega$, basic open sets $O^i_k : k \in m_i$, numbers $l^i_k : k \in m_i$ and bits $b^i_k : k \in m_i$ so that

- β_i is the largest Cantor-Bendixson rank of points in the set $K_n \setminus \bigcup_{j \in i, k \in m_j} O^j_k$, $\{x^i_k : k \in m_i\}$ is the list of these finitely many points of largest possible rank;
- $\{l^i_k : k \in m_i\}$ are numbers increasing with k, larger than $\max\{l^j_k : j \in i, k \in m_j\}$, such that $g_{\omega\alpha}(l^i_k) = \omega\beta_i + k$;
- $\{O^i_k : k \in m_i\}$ are small basic open sets containing the points x^i_k respectively such that all points $x \in O^i_k \cap C$ yield the same value of $h(x)(l^i_k)$, and this value is b^i_k.

Now, since the ordinals β_i decrease with i, the process has to end at some finite stage i_n. Let $O_n = \bigcup_{k\in m_j, j\in i_n} O^i_k$, let $a_n = \{l^i_k : k \in m_j, j \in i_n\}$, and let $u_n : a_n \to 2$ be the map defined by $u_n(l^i_k) = b^i_k$. It is easy to verify that $K_n \subset O_n$, a_n is a $\omega\alpha$-small set and $C \cap O_n \Vdash \dot{y} \cap u_n \neq 0$. Player II responds with the move O_n. It is immediate that the result of the play, the set $\bigcap_n O_n \subset C$, is an I-positive compact set which as a condition in the poset P_J forces that $\forall n\ \dot{y} \cap u_n \neq 0$ and so $\dot{y}$ belongs to the generator of the σ-ideal $I_{\omega\alpha}$ associated with the sequence $\vec{u} = \langle u_n : n \in \omega\rangle$.

□

Since the quotient forcing of the σ-ideal $I_{\omega\alpha}$ forces its generic point not to belong to any $I_{\omega\alpha}$-small set, its overspill ordinal must be at least α and the theorem follows. □

The method of proof of the previous theorem shows that the verification of overspill and other properties of σ-ideals is at least a coanalytic job:

Theorem 4.6. *There is a Polish space Y and a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ family $\{I_y : y \in Y\}$ of σ-ideals on 2^ω σ-generated by closed sets, and a complete coanalytic set $C \subset Y$ such that $y \in C$ iff I_y has overspill iff I_y has hereditary overspill iff P_{I_y} is bounding iff P_{I_y} adds no Cohen real.*

Here, a family I_y is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset Y \times 2^\omega$ the set $\{y \in Y : A_y \in I_y\}$ is coanalytic.

Proof. Let Y be the collection of all linear orders on ω, and let $C \subset Y$ be the collection of all wellorders. This is a complete coanalytic set. Fix a function $\pi : \omega \to \omega$ such that every natural number has infinite π-preimage. For every $y \in Y$, let I_y be the σ-ideal generated by sets $A_{\vec{u}} = \{x \in 2^\omega : \forall n\ \vec{u}(n) \cap x \neq 0\}$, where $\vec{u}$ ranges over all sequences $\vec{u} = \vec{u}(0), \vec{u}(1), \dots$ such that each $\vec{u}(n)$ is a finite partial function from ω to 2 such that $k \in l$ both in $\mathrm{dom}(\vec{u}(n))$ implies that $\pi(l)$ is smaller in the linear ordering y than $\pi(k)$, and moreover $\max\mathrm{dom}(\vec{u}(n)) < \min\mathrm{dom}(\vec{u}(n+1))$. I claim that this collection of σ-ideals works as required.

First of all, if y is a wellorder, then the σ-ideal I_y has the hereditary overspill property, and the quotient P_{I_y} is bounding as proved in the previous theorem. On the other hand, if y is not a wellorder and $a \subset \omega$ is an infinite set such that $k \in l$ both in a implies $\pi(l)$ is less in y than $\pi(k)$, then I will show that P_{I_y} forces $\dot{x}_{gen} \restriction a$ to be a Cohen real. The failure of the overspill property of I_y will immediately follow. And indeed, if $O \subset 2^a$ is an open dense set, then a simple argument will yield partial functions $\vec{u}(n)$ for every $n \in \omega$ so that $\max\mathrm{dom}(\vec{u}(n)) < \min\mathrm{dom}(\vec{u}(n+1))$ and for every $n \in \omega$, every point $z \in 2^a$ disjoint from $\vec{u}(n)$ belongs to O. Since the set $A_{\vec{u}}$ is in the σ-ideal I_y, it must be the case that P_{I_y} forces $\dot{x}_{gen} \restriction \check{a} \in \dot{O}$, and as O was arbitrary, $\dot{x}_{gen} \restriction \check{a}$ is forced to be a Cohen real.

The evaluation of the complexity of the family $\{I_y : y \in Y\}$ is left to the interested reader. □

An illuminating extra piece of information is that the statement that the overspill ordinal is equal to 1 (in other words, Player I has a winning strategy using finite sets only) can be immediately translated into preexisting forcing language.

Definition 4.7. A forcing P is said to have *the weak Sacks property* if for every function $f \in \omega^\omega$ in the P-extension there is a ground model infinite set $a \subset \omega$ and a ground model function g with domain a such that for every $n \in a$, $|g(n)| \leq 2^n$ and $f(n) \in g(n)$.

The weak Sacks property is an obvious weakening of Sacks property which requires $a = \omega$ [1, Definition 6.3.37]. It clearly implies the bounding property, and in a suitably definable case, its conjunction with adding no independent reals is in fact equivalent to the conjunction of the bounding property and P-point preservation [13]. The main point here is

Theorem 4.8. *Let I be a σ-ideal on a Polish space X such that the poset P_I is proper and bounding, and the set $I \cap K(X)$ is coanalytic. Then the following are equivalent:*

1. *P_I has the weak Sacks property;*
2. *every I-positive Borel set has an I-positive compact subset C such that Player I has a winning strategy in the game $G(I, C)$ which uses only finite sets as moves.*

Proof. (2) immediately implies (1). Suppose that $B \in P_I$ is a condition and $\dot{y}$ a name for a point in the Baire space ω^ω. Since the forcing P_I is bounding, there is a compact I-positive set $C \subset B$ and a continuous function $f : C \to \omega^\omega$ such that $C \Vdash \dot{y} = \dot{f}(\dot{x}_{gen})$ and Player I has a winning strategy σ in the game $G(I, C)$ that uses only finite sets as moves. Now consider the counterplay against the strategy σ in which Player II at round n finds a number $m = m_n$ such that $2^m > |K_n|$ and plays an open set O_n covering K_n on which the continuous function $x \mapsto f(x)(m)$ takes fewer than 2^m many values, collected in some set $g(m)$ of size $< 2^m$. In the end, the result of the play is an I-positive compact set $D \subset C$, and, writing $a = \{m_n : n \in \omega\}$, it forces $\forall n \in a\ \dot{y}(n) \in \check{g}(n)$ as desired.

The other direction is more difficult. Suppose that (2) fails below some I-positive Borel set $B \subset X$. Use the bounding property to thin out B if necessary so that all open sets from the countable basis of the space X are relatively clopen in B and B is compact. Since (2) fails, it must be the case that Player II has a winning strategy σ in the game G similar to $G(I, B)$ except Player I is allowed to play finite sets only in the game G. Now, by induction on $n \in \omega$ build increasing finite sets e_n of finite plays of the game G in which Player II follows the strategy σ and, whenever $t \in e_n$ is a play with the last move the strategy σ made in it a certain open set O, whenever $K \subset O \cap B$ is a set of size 2^n then there is a one round extension s of t in the set e_{n+1} such that the last move of strategy σ in s contains K as a subset.

In order to see how to make the induction step, choose $t \in e_n$ and note that the set $(O \cap B)^{2^n}$ is compact, and the set $U = \{P^{2^n}$:there is a move $K \in [O \cap B]^{2^n}$ of Player I that provokes the strategy σ to play $P\}$ covers it, since every set of size 2^n will provoke σ's answer that covers it. A compactness argument will yield a finite subcover of U, which will lead immediately to the construction of the finite set e_{n+1} on the next stage of induction.

Once the induction is complete, consider the function f defined on the set B so that $f(x)(n) =$ the intersection of the collection of those open sets used as last moves of plays in the set e_n to which x belongs. I claim that the name $\dot{f}(\dot{x}_{gen})$ violates the weak Sacks property: there is no condition $C \subset B$, with an infinite set $a \subset \omega$ and a function g on a such that for every $n \in a$, $|g(n)| < 2^n$ and $C \Vdash \dot{f}(\dot{x}_{gen})(n) \in \check{g}(n)$. Suppose for contradiction that such C, a, g exist and thin out C so that for every $x \in C$ and every $n \in a$, $f(x)(n) \in g(n)$. Let $n_i : i \in \omega$ enumerate the set a in an increasing order and by induction on i build plays $t_i \in e_{n_i}$ so that $t_0 \subset t_1 \subset \dots$, t_{i+1} is a one move extension of t_i, and its last move still contains C as a subset. If this succeeds, then in the end the result of the play $\bigcup_i t_i$ contains C as a subset and Player I won, contradicting the assumption that σ was a winning strategy for Player I. The induction step is simple: given t_i, find a set $K \subset C$ of size $2^{n_{i+1}}$ such that the values $f(x)(n_{i+1})$ for $x \in K$ exhaust all possibilities in C. Note that there are fewer than $2^{n_{i+1}}$ possibilities for this value at the set C since they are controlled by the function g. By the construction of the set $e_{n_{i+1}}$, there must be a one round extension t_{i+1} of t_i such that the last move O on it contains K as a subset. But then, O also contains C as a subset: for every point $x \in C$, there is $x' \in K$ such that $f(x)(n) = f(x')(n)$, and by the definition of the function f, $x \in f(x)(n) = f(x')(n) \subset O$ as desired! □

This theorem yields many examples of σ-ideals with the hereditary overspill property, since Sacks or weak Sacks property are fairly common in the realm of definable forcing. Thus, the σ-ideal σ-generated by Borel subsets of 2^ω consisting of pairwise non-modulo-finite-equal sequences has the overspill property, since the quotient forcing is proper and has the Sacks property [12, Section 4.7.1].

5 The σ-ideal generated by H-sets

In this section, I will apply the work of previous section to obtain an independence result for two σ-ideals from harmonic analysis. Let $\mathbb{T}$ be the unit circle, understood as the group $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 5.1. A set $A \subset \mathbb{T}$ is a *set of uniqueness* if every trigonometric series converging to zero pointwise off A is trivial. U_σ is the σ-ideal σ-generated by closed sets of uniqueness.

Fourier showed that the empty set is a set of uniqueness, and Cantor proved that every countable closed set is a set of uniqueness. While it is not true that the union of two sets of uniqueness is a set of uniqueness, and it is not known whether this holds for Borel sets, Bary [2] showed that the union of countably many *closed* sets of uniqueness is a set of uniqueness.

Definition 5.2. A set $A \subset \mathbb{T}$ is an *H-set* if there is an infinite set $b \subset \omega$ and a nontrivial open set $O \subset \mathbb{T}$ such that for every $n \in b$, $nA \cap O = 0$. H_σ is the σ-ideal σ-generated by H-sets.

Rajchman [9] defined H-sets in a search for perfect sets of uniqueness. He proved that H-sets are sets of uniqueness, and since the closure of an H-set is again an H-set, it follows that $H_\sigma \subset U_\sigma$. He also showed that the Cantor middle third set is an H-set, producing a perfect set of uniqueness. The combinatorics of both H-sets and sets of uniqueness is quite complicated [5]. I will show

Theorem 5.3. *Suppose that the Generalized Continuum Hypothesis holds and $\kappa \geq \aleph_1$ is a regular cardinal. Then there is a cardinal preserving forcing extension in which $\mathtt{cov}(U_\sigma) = \aleph_1$ and $\mathtt{cov}(H_\sigma) \geq \kappa$.*

Proof. The plan of attack is straightforward. Consider the quotient forcing P_{H_σ} of Borel H_σ-positive sets ordered by inclusion. H_σ turns out to be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal σ-generated by closed sets, with hereditary overspill property; the poset P_{H_σ} is bounding and preserves Baire category. Consider the countable support product P of κ many copies of P_{H_σ}. This is a proper bounding forcing preserving Baire category; a standard argument shows that it is $\aleph_2$-c.c. and therefore preserves all cardinals. It is not difficult to show that $P \Vdash \mathtt{cov}(H_\sigma) \geq \kappa$, since any H-set in the extension can cover at most countably many among the κ many P_{H_σ}-generic points added by the product. Most importantly, a result of Loomis [7] shows that there is an analytic (in fact $G_{\delta\sigma}$) set $A \subset K(\mathbb{T})$ that contains all countable closed sets and consists only of sets of uniqueness; in other words, the σ-ideal U_σ does not have the overspill property. By Theorem 3.7, in the P-extension, $\mathbb{T} = \bigcup(A \cap V)$ and therefore $\mathtt{cov}(U_\sigma) = \aleph_1$.

In order to fill in the details of this plan, I must first prove the requisite properties of the poset P_{H_σ}. To do that, I will consider a different poset that seems to have nothing to do with harmonic analysis. Let $\omega = \bigcup_k a_k$ be a partition into infinite sets, and let I be the σ-ideal σ-generated by sets $A_g = \{x \in 2^\omega : g \subset x\}$ as g ranges over all infinite partial functions from ω to 2 with domain included in one of the sets a_k for some $k \in \omega$. The first sentence of the following claim is proved essentially by a word-by-word repetition of arguments in Theorem 4.1, for the case $\alpha = 1$. The second sentence follows from the first by the general facts about definable forcing, such as [12, Theorems 4.1.2 and 3.3.2]

Claim 5.4. *I is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal with the hereditary overspill property, it is generated by closed sets and every analytic I-positive set has a compact I-positive subset. Thus, the quotient forcing is proper, preserves Baire category, and it is bounding.*

The quotient poset P_I is quite complicated, in particular, it seems to be highly inhomogeneous. It can be combinatorially presented as the poset of those binary trees $T \subset 2^{<\omega}$ such that for every node $t \in T$ and every number $k \in \omega$ there is a number $m \in \omega$ such that for every $n \in a_k$ greater than m there are extensions s_0, s_1 of t in the tree T such that $s_0(n) = 0$ and $s_1(1) = 1$. This should be compared to the poset introduced by Shelah [10, Proposition 1.10] and studied also by Spinas [11].

To see the connection between the poset P_I and P_{H_σ}, enumerate the nontrivial rational open intervals of $\mathbb{T}$ by $\{O_k : k \in \omega\}$ and consider the map $h : \mathbb{T} \to 2^\omega$ defined by $h(x)(m) = 0$ if $nx \in O_k$, where m is the n-th element of a_k. It is immediate that h is a one-to-one Borel function, thus its range $\text{rng}(h) \subset 2^\omega$ is a Borel set, and the function h also carries the σ-ideal H_σ to $I \restriction \text{rng}(h)$. It immediately follows that the σ-ideal H_σ has all the properties of I claimed in the above claim.

The rest of the plan outlined in the first paragraph follows from the references there, with perhaps one exception–that any H-set in the P-extension contains only countably many P_{H_σ}-generic points. This is also a standard argument using the rectangular Ramsey property of H_σ. Note that the above Claim and [12, Theorem 5.2.6] show that the rectangular Ramsey property indeed holds for the σ-ideal H_σ in any countable number of dimensions.

Suppose that $p \Vdash \dot{A}$ is an H-set, for some condition $p \in P$. By the standard analysis of the countable support product of definable forcing, there is a countable set $b \subset \kappa$, H_σ-positive sets $\{K_\alpha : \alpha \in b\}$ and a Borel set $D \subset \Pi_{\alpha \in b} K_\alpha \times \mathbb{T}$ such that the vertical sections of D are all H-sets and the condition $q \leq p$, $q = \Pi_{\alpha \in b} K_\alpha$ forces $\dot{A} \subset D_{\vec{x}_{gen} \restriction b}$. I claim that the only generic points in the product that can belong to the set $\dot{A}$ are indexed by the ordinals in the set b. Indeed, choose a condition $r \leq q$ and an ordinal $\gamma \in \kappa \setminus b$; thinning out if necessary I may find a countable set $c \supset b \cup \{\gamma\}$ and H_σ-positive sets $\{L_\alpha : \alpha \in c\}$ such that $r = \Pi_{\alpha \in c} L_\alpha$. The set $E \subset \Pi L_\alpha$, $E = \{\vec{x} : \vec{x}(\gamma) \in D_{\vec{x} \restriction b}\}$ is Borel and has H_σ-small sections in γ-th coordinate. Therefore, it cannot contain a Borel rectangular box with I_α-positive sides, and by the rectangular Ramsey property, it must be the complement of E that contains such a box $s = \Pi_{\alpha \in c} M_\alpha$. The condition $s \leq r$ must force the γ'th generic point not to belong to $\dot{A}$, as an immediate absoluteness argument shows.

□

As the last point in the paper, I will prove two independence results complementary to Theorem 5.3. They show that there is a great degree of freedom in moving the covering numbers of the σ-ideals mentioned around by forcing.

Theorem 5.5. *It is consistent with ZFC that $\mathbb{T}$ is covered with $\aleph_1$ many H-sets while the continuum is very large.*

Proof. It is enough to reach for a model of ZFC in which the continuum is large while there is a P-point basis of size $\aleph_1$, such as in the product Sacks extension. For every point $x \in \mathbb{T}$ there is a set $a \subset \omega$ in the P-point ultrafilter such that the points $\{nx : n \in a\}$ converge, and therefore they avoid a certain nonempty open interval in the circle $\mathbb{T}$. This shows that the $\aleph_1$ many sets $B_{a,O} = \{x \in \mathbb{T} : \forall n \in a\ nx \notin O\}$, as a ranges over the P-point basis of size $\aleph_1$

and O ranges over all possible open intervals with rational endpoints, cover the circle $\mathbb{T}$, and they are all H-sets. □

Theorem 5.6. *It is consistent with ZFC that $\mathbb{T}$ cannot be covered by fewer than $\aleph_2$ many closed sets of uniqueness while there are dominating, nonmeager and nonnull sets of size $\aleph_1$.*

Proof. Consider the σ-ideal U_0 of sets of extended uniqueness on $\mathbb{T}$. The deep results of Debs and Saint-Raymond [3] show that this is a σ-ideal σ-generated by closed sets and it is polar. The collection of closed sets in U_0 is coanalytic, in fact $\mathbf{\Pi}^1_1$-complete by a result of Solovay and Kaufman [5, Section IV.2], and so by [12, Theorem 3.8.9] the σ-ideal U_0 is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Thus, the quotient P_{U_0} is proper, bounding, preserves Baire category, and outer Lebesgue measure by [12, Theorem 3.6.2]. Moreover, every set of uniqueness is a set of extended uniqueness, and so the poset P_{U_0} forces its generic real not to belong to any ground model coded closed sets of uniqueness. Ergo, starting with a model of the Continuum Hypothesis and iterating P_{U_0} ω_2 many times, a model of the statement of the theorem is achieved as the various preservation theorems of [12, Section 6.3] or [1, Section 6.3] show.

Note that the poset does not have the Sacks property–by the results of the previous section, it would imply a particularly strong version of overspill, and the σ-ideal U_0 does not have the overspill property. Thus, in the extension, the cofinality of the Lebesgue null ideal is $\aleph_2$. I do not know if the products of the poset P_{U_0} preserve outer Lebesgue measure, and therefore I do not know if it is possible to push the continuum beyond $\aleph_2$. □

References

[1] Tomek Bartoszynski and Haim Judah. *Set Theory. On the structure of the real line.* A K Peters, Wellesley, MA, 1995.

[2] N. K. Bary. *A treatise on trigonometric series.* Macmillan, New York, 1964.

[3] G. Debs and J. Saint-Raymond. Ensembles boreliens d'unicité at d'unicité au sens large. *Ann. Inst. Fourier (Grenoble)*, 37:217–239, 1987.

[4] Thomas Jech. *Set Theory.* Academic Press, San Diego, 1978.

[5] Alexander Kechris and Alain Louveau. *Descriptive set theory and the structure of sets of uniqueness.* Cambridge University Press, Cambridge, 1989.

[6] Alexander S. Kechris. *Classical Descriptive Set Theory.* Springer Verlag, New York, 1994.

[7] L. Loomis. The spectral characterization of a class of almost periodic functions. *Annals of Mathematics*, 72:362–368, 1960.

[8] Jan Pelant and Miroslav Zelený. The structure of the sigma-ideal of sigma-porous sets. *Commentationes Mathematicae Universitatis Carolinae*, 45:37–72, 2004.

[9] Alexander Rajchman. Sur l'unicité du développement trigonométrique. *Fundamenta Mathematicae*, 3:287–302, 1922.

[10] Saharon Shelah. Vive la différence i: Nonisomorphism of ultrapowers of countable models. In *Set Theory of the Continuum*, volume 26 of *Mathematical Sciences Research Institute Publications*, pages 357–405. Springer Verlag, 1992. math.LO/9201245.

[11] Otmar Spinas. Analytic countably splitting families. *Journal of Symbolic Logic*, 69:101–117, 2004.

[12] Jindřich Zapletal. *Forcing Idealized*. Cambridge Tracts in Mathematics 174. Cambridge University Press, Cambridge, 2008.

[13] Jindřich Zapletal. Preservation of P-points and definable forcing. *Fundamenta Mathematicae*, 204:145–154, 2009.