



Representations of Matroids in Semimodular Lattices

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Representations of matroids in semimodular lattices and Coxeter matroids in chamber systems are considered in this paper.

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INTRODUCTION

This paper is devoted mostly to the explanation of the equivalence of two definitions of representability of matroids: classical, in terms of vector configurations; and a more general definition, in terms of retractions of buildings of type A_n , as it arises in the theory of Coxeter matroids [3]. We wish to generalize further both concepts and consider, therefore, representations of matroids in semimodular lattices and Coxeter matroids in chamber systems. Definitions and notation used in the paper are mostly standard. Those related to Coxeter groups and buildings can be found in [11, 13, 15], to lattices and matroids in [18]. See also the forthcoming book [5].

1. SEMIMODULAR LATTICES AND MATROIDS

We expect this paper to be read both by algebraists and combinatorialists. This section contains definitions that will be familiar to the latter but perhaps not to the former.

Lattices. We recall some standard definitions concerning partially ordered sets and lattices [18, Chapter 3].

Let \leq be a partial ordering of the set X . We set $x < y$ if $x \leq y$ and $x \neq y$. An element x covers element y if $y < x$ and if $y \leq z \leq x$, then either $y = z$ or $z = x$. If X has a unique minimal element 0 , then an *atom* is an element which covers 0 . Given two elements $x, y \in X$, the *interval* $[x, y]$ is the set $\{z \mid x \leq z \leq y\}$. A finite *chain* is a totally ordered subset $x_0 < x_1 < \cdots < x_k$, and the *length* of this chain is k . If X has the minimal element 0 , then the *height* $h(x)$ is the maximum of lengths of chains $0 < x_1 < \cdots < x_k = x$ between 0 and x , provided this maximum exists.

A partially ordered set X satisfies the *Jordan–Hölder condition* if, given any two elements x and y in X , all maximal chains between x and y have the same (finite) length.

A *lattice* is a partially ordered set L such that L contains, with any two elements x and y , their *least upper bound* $x \vee y$ and *greatest lower bound* $x \wedge y$. The elements $x \vee y$ and $x \wedge y$ are also called, correspondingly, the *join* and *meet* of x and y . In this paper, all lattices are assumed to have finite height, and therefore a unique maximal element 1 . A lattice L is *semimodular* if, for any elements x and y in L covering their meet $x \wedge y$, their join $x \vee y$ covers both x and y . Equivalently, a lattice L is semimodular if and only if it satisfies the Jordan–Hölder condition and its height function satisfies the *semimodular inequality*

$$h(x) + h(y) \geq h(x \wedge y) + h(x \vee y).$$

It is worthwhile to recall at this point that *geometric lattices*, i.e., semimodular lattices of finite height in which every element is a join of atoms, are exactly lattices of flats of matroids.

Bases. We say that a set of atoms $A = \{a_1, \dots, a_k\}$ in L is *independent* if

$$a_{i_0} \wedge (a_{i_1} \vee \dots \vee a_{i_m}) = 0$$

for any set of pairwise distinct atoms $a_{i_0}, a_{i_1}, \dots, a_{i_m}$. A *basis* $B = \{b_1, \dots, b_n\}$ in L is an independent set of atoms such that $b_1 \vee \dots \vee b_n = 1$. Obviously, in this situation $n = h(L)$ and

$$0 < b_1 < b_1 \vee b_2 < \dots < b_1 \vee \dots \vee b_n$$

is a maximal chain in L ; we say in this situation that the chain is *spanned* by the (ordered) basis b_1, \dots, b_n .

The following observation is immediate.

PROPOSITION 1. *A semimodular lattice of finite height is geometric if and only if every maximal chain is spanned by some ordered basis.*

Closure systems and matroids. A *closure operator* on a set E is an increasing, monotone, idempotent function

$$\tau : 2^E \longrightarrow 2^E$$

on the set 2^E of all subsets in E . This means that for all $A, B \subseteq E$:

- (1) $A \subseteq \tau(A)$;
- (2) $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$;
- (3) $\tau(\tau(A)) = \tau(A)$.

A *matroid* $M = (E, \tau)$ is a finite set E with a closure operator τ satisfying the *Exchange Property for Closure Operators*:

$$\text{If } x, y \notin \tau(A) \quad \text{and} \quad y \in \tau(A \cup \{x\}), \text{ then } x \in \tau(A \cup \{y\}).$$

Bases of a matroid. A set $A \subseteq E$ is called *independent* if $x \notin \tau(A \setminus \{x\})$ for all $x \in A$. Maximal independent sets in E exist by Zorn's Lemma and are called *bases* of M , and, vice versa, given the set \mathcal{B} of bases of a matroid on a finite set E , we can define *independent* sets as subsets of bases, and for an arbitrary subset $X \subseteq E$, *rank* $r(X)$ as the maximum of the cardinalities of independent subsets in X . Then we define the closure operator τ on E as

$$\tau(X) = \{y \in E \mid r(X) = r(X \cup \{y\})\}.$$

It is well known that τ is the closure operator of a matroid on E . Thus we can define a matroid starting either with the closure operator or the bases, and these are equivalent definitions, as may be seen in any standard text. Closed sets in G are called *flats* of the matroid \mathcal{B} . It is well known that the flats of a matroid form a geometric lattice.

2. REPRESENTATIONS OF MATROIDS IN SEMIMODULAR LATTICES

Let \mathcal{B} be a matroid on the set E with the lattice of flats F . An order-preserving injective map f from F to a lattice L is called a *representation* of F (or of \mathcal{B}) if it preserves joins, $f(x \vee y) = f(x) \vee f(y)$.

THEOREM 2. *Let L be a lattice and B a basis in L . Fix some element $x \in L$. Denote, for a subset $A = \{a_1, \dots, a_m\}$ in B ,*

$$\tau(A) = \{b \in B \mid b \leq x \vee a_1 \vee \dots \vee a_m\}.$$

Then τ is the closure operator for a matroid M on the set B . The map

$$\tau(\{a_1, \dots, a_m\}) \mapsto x \vee a_1 \vee \dots \vee a_m$$

is a representation of the lattice of flats of M into $[x, 1]$.

PROOF. Obviously τ is a closure operator, and we only need to check the Exchange Property. Let $A \subseteq B$, $b, c \in B$, b and c do not lie in $\tau(A)$ and $c \in \tau(A \cup \{b\})$. Denote by a the join of all elements in A and set $y = x \vee a$, then $y \vee b > y$ and $c \vee y \vee b = y \vee b$. Since c and b are atoms, we have, by the semimodular inequality,

$$h(y) < h(y \vee b) \leq h(b) + h(y) - h(y \wedge b) = 1 + h(y).$$

Therefore $h(y \vee b) = h(y) + 1$. Analogously $h(y \vee c) = h(y) + 1$. Since $c \in \tau(A \cup \{b\})$, $c \leq y \vee b$ and $y \leq y \vee c \leq y \vee b$. Comparing the heights, we have $y \vee b = y \vee c$, which means $b \in \tau(A \cup \{c\})$. This proves the Exchange Property for τ . Now it is obvious that the map

$$\tau(\{a_1, \dots, a_m\}) \mapsto x \vee a_1 \vee \dots \vee a_m$$

is a join-preserving map of M in $[x, 1]$. The injectivity follows by noting that a maximal chain in the lattice of flats of M must have been mapped by τ to a maximal chain in $[x, 1]$. \square

We shall denote by $M_{x,B}$ the matroid on B constructed in Theorem 2 for an element $x \in L$ and the basis B . We say that this matroid is *represented* by x in L with respect to B .

Let L be a semimodular lattice of finite height n with a basis $B = [n]$. Let $x \in L$. Note that $M_{x,B}$ is now a matroid on $[n]$. We wish to show that there is another way to associate a matroid on $[n]$ with x , which leads directly to the definition of a matroid in terms of the maximality property [6]. If w is a permutation of the set $[n]$, then we assign weights to the elements of the set $[n]$ so that i has greater weight than j if and only if $w^{-1}(i) > w^{-1}(j)$. We say that A is the w -maximal basis of a given matroid on $[n]$ if A is the basis selected by the greedy algorithm with respect to the weights assigned (see any matroid text).

For every permutation w of the set $[n] = \{1, \dots, n\}$ denote by A_w the subset of $[n]$ formed by those elements $w(i) \in [n]$ for which

$$w(1) \vee w(2) \vee \dots \vee w(i-1) \vee x = w(1) \vee w(2) \vee \dots \vee w(i) \vee x. \quad (1)$$

THEOREM 3. *With the notation above,*

$$\mathcal{B}^* = \{A_w \mid w \in \text{Sym}_n\}$$

is the collection of bases of the matroid $M^ = M_{x,B}^*$ on $[n]$, dual to the matroid $M = M_{x,B}$ constructed in the previous theorem. The rank of the matroid M^* equals the height of the element x in the lattice L . Moreover, A_w is the w -maximal element of \mathcal{B}^* .*

PROOF. We shall prove that the collection \mathcal{B}^* is the collection of bases of the matroid M^* and that A_w is the w -maximal element of \mathcal{B}^* .

Define a mapping $\phi : B \rightarrow L' \subseteq [x, 1]$, where $\phi(a) = a \vee x$ and L' is the sublattice of $[x, 1]$ generated by the image of ϕ . Note that $\phi(a)$ is either $0_{L'}$ or an atom of L' for each

$a \in B$, and L' is generated by $\{\phi(a) : \phi(a) \neq 0_{L'}\}$, so L' is a geometric lattice. Furthermore, since B spans 1_L , we have $1_L = 1_{L'}$ and $\text{rank}(L') = n - \text{rank}(x)$. Let $M = M_{x,B}$ denote the matroid on the set B corresponding to the geometric lattice L' (as in Theorem 2), such that each atom b of L' corresponds to the closed set $\{i : \phi(i) \leq b\}$ of M . Then

$$\begin{aligned} w(i) \in A_w &\Leftrightarrow w(i) \leq x \vee w(1) \vee w(2) \vee \cdots \vee w(i-1) \\ &\Leftrightarrow \phi(w(i)) \leq \phi(w(1)) \vee \phi(w(2)) \vee \cdots \vee \phi(w(i-1)) \end{aligned}$$

in L' , which is true if and only if $w(i)$ is dependent on (i.e., is in the closure of) the set

$$\{w(1), w(2), \dots, w(i-1)\}$$

in M . This in turn is true if and only if $w(i)$ is not an element of the w -minimal basis of M , or equivalently, $w(i)$ is an element of the w -maximal basis of M^* . Thus the bases of M^* are precisely the sets A_w , and the basis A_w is w -maximal in \mathcal{B}^* . \square

A flag F of rank (k_1, k_2, \dots, k_m) is a strictly increasing sequence of sets $F^1 \subset F^2 \subset \cdots \subset F^m \subseteq [n]$ with F^i of cardinality k_i for all i . The set F^i is called the i th constituent of F . A collection \mathcal{F} of flags of rank (k_1, \dots, k_m) is called a flag matroid on $[n]$ if for each i , the collection of i th constituents forms the collection of bases of a matroid M_i , called the i th constituent matroid and, furthermore, M_i is a quotient (or strong map image using the identity map on $[n]$) of M_{i+1} for all $i \leq m-1$. See [12] for definitions. We prove in [6] that \mathcal{F} is a flag matroid if and only if for each permutation w of $[n]$, there exists a w -maximal flag in \mathcal{F} , which is, in each constituent, the w -maximal basis.

Now note that if $x_1 < x_2 < \cdots < x_k$ is a chain in L , then, applying the same procedure as in the previous theorem for each x_i , we assign to every permutation $w \in \text{Sym}_n$ the w -maximal flag of bases $A_w^1 \subset A_w^2 \subset \cdots \subset A_w^k$ (it is obvious from the definition of A_w that $A_w^i \subset A_w^{i+1}$). We shall denote this flag by

$$x_w = \{x_w^1 < x_w^2 < \cdots < x_w^k\}.$$

If \mathcal{F} is the set of all flags x_w for $w \in \text{Sym}_n$, then x_w is the w -maximal flag in \mathcal{F} . Hence \mathcal{F} is a flag matroid [6].

Note, finally, that when L is the lattice of subspaces of a finite dimensional vector space V over a division ring D , we come to the classical concept of matroid representation over D . It is well known that in this case representability of a matroid M over D is equivalent to the representability over D of the dual matroid M^* . This could probably best be explained by the fact that the lattice L is modular. In our more general situation of a semimodular lattice L we have two different ways of representing a matroid M in L : one, given by Theorem 2, works for the matroid M ; another one, given by Theorem 3, works for the set of bases of the dual matroid M^* . We cannot, in general, switch the roles of M and M^* . Fortunately, there is no distinction between the dual matroids M and M^* when we look at them from a more general view point of Coxeter matroids.

3. EVERY FLAG MATROID IS REPRESENTABLE

Now we show that every flag matroid may be represented in a semimodular lattice, and even slightly better, in a geometric lattice.

THEOREM 4. *Let \mathcal{F} be a flag matroid map on $[n]$. Then there exists a geometric lattice L of rank n such that each constituent of \mathcal{F} is representable in L , all with respect to the same basis B .*

PROOF. Let \mathcal{F} be a flag matroid on $[n]$. Then \mathcal{F} has constituents which are ordinary matroids M_1, \dots, M_m where M_i is a quotient of M_{i+1} for each i . Without loss of generality, we may assume that \mathcal{F} has constituents of every rank from 1 to n , by inserting Higgs' lifts of M_i toward M_{i+1} where necessary, see [12]. Then by the Factorization Theorem of Strong Maps (see [12, Theorem 8.27]), each strong map $M_{i+1} \rightarrow M_i$ may be realized by the extension of M_{i+1} by a single element, followed by the contraction of that element. It is well known in matroid theory (see [12, Exercise 8.14, p. 245]) that we can realize the entire sequence of strong maps by adjoining $n - 1$ elements z_1, z_2, \dots, z_{n-1} to M_n , to obtain a matroid M^+ in such a way that

$$M_i = M^+ \setminus \{z_1, \dots, z_i\} / \{z_{i+1}, \dots, z_{n-1}\}.$$

Now let L be the geometric lattice of the matroid M^+ . It is now routine to verify that the matroid $M_{x,[n]}$, where $x = z_1 \vee \dots \vee z_i$, is M_j . \square

It is not true that there exists a single geometric lattice L such that all flag matroid maps on $[n]$ are representable in L by flags of matroids of the form $M_{x,B}$ for fixed B (although this is trivial to do if we allow variable B). To see this, we note that there exist *incompatible extensions* according to [9]; that is, extensions of a given matroid which cannot exist in a single common extension. The strong maps which correspond to these extensions can then be placed in sequences of strong maps which give two flag matroids which cannot be represented in a semimodular lattice L with respect to the same basis B .

4. GROUP METRIC ON CHAMBER SYSTEMS

Chamber systems. Our exposition of chamber systems follows [13]. We freely use terminology of the theory of Coxeter groups [7, 13].

A set C is a *chamber system* over a set I if each element i of I determines a partition of C , two elements in the same part being called *i -adjacent*. Thus i -adjacency is an equivalence relation on C . The classes of i -adjacency are called *panels of type i* and the elements of C are called *chambers*. If π is a panel and x is a chamber in π , we shall say, abusing the language, that π is a panel of x . For i -adjacent chambers x and y we shall write $x \sim_i y$.

A *gallery* is a finite sequence of chambers (c_0, \dots, c_k) such that c_{j-1} is adjacent to c_j for each $1 \leq j \leq k$; k is called the *length* of the gallery. The gallery is said to be of *type $i_1 i_2 \dots i_k$* if c_{j-1} is i_j -adjacent to c_j . A gallery (c_0, \dots, c_k) , connecting $x = c_0$ and $y = c_k$, is called a *geodesic gallery* if there is no gallery of strictly smaller length with the same property.

A *morphism* $\phi : C \rightarrow D$ between two chamber systems over the same indexing set I is a map preserving the i -adjacency for each $i \in I$ (thus if x and y are i -adjacent, then $\phi(x)$ and $\phi(y)$ are too).

Coxeter complexes. An important class of examples of chamber systems is provided by Coxeter complexes.

Let W be a Coxeter group with the distinguished set of generators $R = \{r_i \mid i \in I\}$. Take the elements of W as chambers and for each $i \in I$, define i -adjacency by

$$w \sim_i w, \quad wr_i.$$

This gives a chamber system over I , which is called the *Coxeter complex* of W .

W-metric. A gallery $\Gamma = (x_0, \dots, x_k)$ of type $i_1 \cdots i_k$ in W is called *reduced*, if $r_{i_1} \cdots r_{i_k}$ is a reduced expression for an element in W .

Let W be a Coxeter group with the set of standard generators r_1, \dots, r_n .

A *W-metric* on a set C is a map

$$\pi : C \times C \longrightarrow W$$

satisfying, for all $c, d \in C$, the following four properties:

- (1) $\pi(c, d) = \pi(d, c)^{-1}$;
- (2) $\pi(c, d) = 1$ if and only if $c = d$.

If we define two elements $a, b \in C$ to be *r_i -adjacent* if $\pi(a, b) \in \{1, r_i\}$, this turns the set C with a *W-metric* π into a chamber complex of type $R = \{r_1, \dots, r_n\}$. In particular, we shall call elements of C *chambers*. This also allows us to define, as above, galleries and geodesic galleries in C .

The remaining two properties of a *W-metric* π are:

- (3) if $\pi(d, d') = r_i$ is a standard generator and $\pi(c, d) = w$, then $\pi(c, d') \in \{w, wr_i\}$;
- (4) let $c = c_0, c_1, \dots, c_l = d$ be a geodesic gallery connecting c and d . Assume that c_{j-1} and c_j are *r_{i_j} -adjacent*, $j = 1, \dots, l$. Then

$$\pi(c, d) = r_{i_1} \cdots r_{i_l},$$

and the term $r_{i_1} \cdots r_{i_l}$ is reduced.

In this situation we shall say that C is a *chamber complex with a W-metric*.

Note that the group W itself has a natural *W-metric* $\pi(u, v) = u^{-1}v$. If A and B are two sets with a *W-metric*, an *W-isometry* from A to B is a map preserving the *W-metric*. Note that the left multiplication by an element of W is a *W-isometry* of W .

Buildings. This is the most prominent class of chamber complexes with a *W-metric*.

Let W be a Coxeter group with the distinguished set of generators $R = \{r_i, i \in I\}$. By definition, a *building of type W* is a chamber system Δ with *W-metric* such that each panel belongs to at least two chambers, and such that if $w = r_{i_1} \cdots r_{i_k}$ is a reduced expression for $w \in W$, then $\pi(x, y) = w$ if and only if x and y can be joined by a gallery of type $i_1 \cdots i_k$.

The Coxeter complexes are buildings, with a *W-metric* $\pi(x, y) = x^{-1}y$.

In the special case when $W = \text{Sym}_n$ is the symmetric group on $n \geq 4$ letters, buildings of type W , or type A_n as they are usually called, are flag complexes of projective spaces over division rings [15]. Therefore buildings of type A_n are a special case of chain complexes of semimodular lattices of finite height.

5. ABELS' COMPLEX FOR A SEMIMODULAR LATTICE

Herbert Abels [2] has shown that the chain complex of a semimodular lattice of finite height n has a natural structure of a chamber complex with Sym_n -metric.

We shall use the following definition from Abels' work. Let L be a semimodular lattice of finite height n and C the chamber complex of maximal chains in L . Let

$$c = \{c_0 < c_1 < \cdots < c_n\}$$

and

$$d = \{d_0 < d_1 < \cdots < d_n\}$$

be two chambers in C . The *Jordan–Hölder permutation* $\pi(c, d) \in \text{Sym}_n$ can be defined for $i > 0$ by the following formula:

$$\pi(c, d)(i) = j \quad \text{if} \quad \begin{cases} d_{i-1} \vee c_k = d_i \vee c_k & \text{for all } k \geq j \\ d_{i-1} \vee c_k < d_i \vee c_k & \text{for all } k < j. \end{cases} \quad (2)$$

It is shown in [1] that $\pi(c, d)$ is a permutation of $[n]$. Moreover, it follows from [2, Theorem 3.5(a)–(e)] that π satisfies conditions (1)–(4). Thus C is a chamber complex with a Sym_n -metric. Note that for Sym_n considered as a Coxeter group, the standard generators are the adjacent transpositions, $r_i = (i, i + 1)$. One may check that c and d are i -adjacent if and only if $c_j = d_j$ for all $j \neq i$.

6. APARTMENTS AND RETRACTIONS

From now on we assume that we have a chamber system C with a W -metric π .

Apartments. A π -isometric image of the Coxeter complex W in C is called an *apartment*.

Note that this condition is satisfied when C is the chain complex of the lattice of flats of a matroid \mathcal{B} : if B is a basis in \mathcal{B} , the chains spanned by all orderings of B form an apartment.

In another important class of examples, namely buildings, apartments exist by Theorem 3.6 in [13]. Moreover, by Corollary 3.7 in [13], any two chambers in a building lie in a common apartment.

Retractions. Assume that C has an *apartment* A . Then, for any given chamber $a \in A$, we can define the *retraction* of C onto A with the *centre* a as the map

$$\rho_a : C \longrightarrow A$$

which sends the chamber $c \in C$ to the unique chamber $b \in A$ with the property $\pi(a, c) = \pi(a, b)$. Note that retractions have two properties:

- ρ_a sends adjacent chambers to adjacent chambers, by (3), and thus maps galleries in C onto galleries in A ;
- ρ_a sends geodesic galleries starting at a to geodesic galleries, by (4).

Matroid maps. Let \mathcal{F} be a flag matroid of rank $(1, 2, 3, \dots, n)$ on $[n]$. Let

$$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

be a permutation of $[n]$. We think of w as determining a linear order $a_1 < a_2 < \cdots < a_n$ on $[n]$, and let $F_1^w \subseteq F_2^w \subseteq \cdots \subseteq F_n^w$ be the flag of w -maximal bases. Let $\{b_i\} = F_i^w \setminus F_{i-1}^w$ for each i . Then

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

is a permutation, which we denote $\mu(w)$. Now, μ is a map from Sym_n to Sym_n , which is called the *matroid map*. It has the property that

$$w^{-1}\mu(u) \leq w^{-1}\mu(w)$$

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for all $u, w \in \text{Sym}_n$, where \leq denotes the well-known Bruhat order on Sym_n (see [3]). Essentially, this states that $\mu(w)$ is as far from w as possible, subject to being a permutation which corresponds to some flag in \mathcal{F} . Furthermore, the matroid map μ characterizes the flag matroid \mathcal{F} .

Incidentally, we remark that similar results are true for sparser flag matroids (that is, those whose ranks are a proper subsequence of $(1, 2, 3, \dots, n)$), including the case of a single matroid. However, in these cases the matroid map, while still defined on Sym_n , has as its codomain the set of left cosets of Sym_n by an appropriate parabolic subgroup.

The matroid map from a retraction. Now let us identify the apartment A with W and, given a chamber $c \in C$, define the map $\mu_c : W \rightarrow W$ by the rule

$$\mu_c(w) = \rho_w(c).$$

THEOREM 5. For each $c \in C$, μ_c is a matroid map, i.e.,

$$w^{-1}\mu_c(u) \leq w^{-1}\mu_c(w)$$

for all $u, w \in W$.

PROOF. This follows immediately from [3, Theorem 5]. The proof is so simple, though, that it can be summarized here as well. Let Γ be a geodesic gallery connecting the chambers $w \in W$ and c , Γ' its image under the retraction ρ_w and Γ'' the image of Γ under the retraction ρ_u . Then Γ' is a geodesic gallery connecting w and $\rho_w(c) = \mu_c(w)$ and Γ'' is a gallery connecting w and $\rho_u(c) = \mu_c(u)$. Note now that Γ' and Γ'' , being the images of Γ under the chamber morphisms ρ_w and ρ_u , are of the same type. This situation is exactly the geometric interpretation of the Bruhat ordering and gives the desired inequality

$$w^{-1}\mu_c(u) \leq w^{-1}\mu_c(w). \quad \square$$

EXAMPLE. Consider the semimodular lattice illustrated in Figure 1. We leave it to the reader to check that it is indeed semimodular. The reader should also check using Eqn. (2) that for the chambers (maximal chains) shown, $\pi(a, c) = \pi(a, b) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$. But

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = (1, 2)(2, 3)(1, 2)(3, 4)(2, 3) = r_1 r_2 r_1 r_3 r_2.$$

The reader should find the geodesic gallery of type 1, 2, 1, 3, 2 between a and c , and similarly between a and b . Note that $\mu_c(a) = b$, or

$$\mu_c \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}.$$

Thus $\{4\}$, $\{2, 4\}$, $\{1, 2, 4\}$, $[4]$ must be the the flag of maximal bases for the ordering $3 < 1 < 4 < 2$ in the flag matroid corresponding to μ_c .

7. THE EQUIVALENCE OF THE TWO DEFINITIONS

Again let L be a semimodular lattice and C its chamber complex of maximal chains. Let A be the chamber complex of maximal chains of the lattice generated by a basis a_1, \dots, a_n in L . Obviously A is isomorphic to the complex of flags of subsets in $[n]$. Also it is easy to

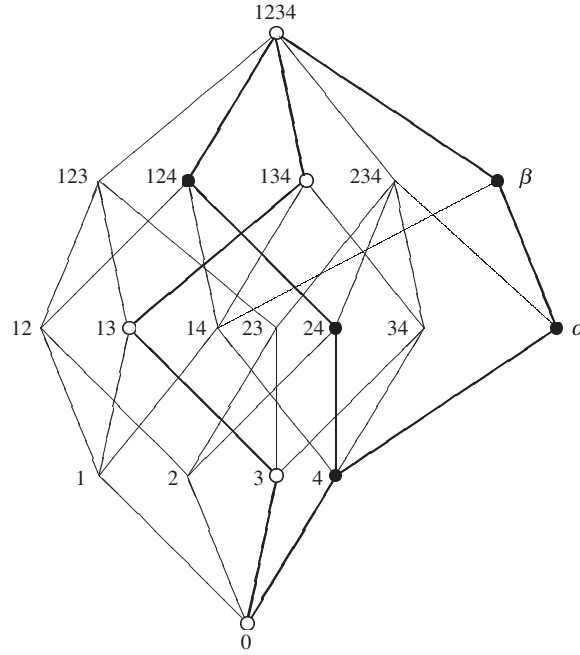


FIGURE 1. If a is the maximal chain $0 < 3 < 13 < 134 < 1234$, b is $0 < 4 < 24 < 124 < 1234$, and c is $0 < 4 < \alpha < \beta < 1234$, check that both $\pi(a, c)$ and $\pi(a, b)$ are the permutation $1 \mapsto 3, 3 \mapsto 2, 2 \mapsto 4, 4 \mapsto 1$. Thus b is the retraction of c with centre a onto the apartment A , where A is the complex of all maximal chains determined by some permutation of $[4]$.

see that A is the Coxeter complex for Sym_n as a Coxeter group. Thus A is an apartment, and we identify it with Sym_n . Indeed, we will again identify a_i with i . Specifically, we identify $w \in \text{Sym}_n$ with the flag of sets

$$w([1]) \subset w([2]) \subset \cdots \subset w([n]).$$

Let $x = \{0 < x_1 < x_2 < \cdots < x_n = 1\}$ be a maximal chain in L and let $w \in \text{Sym}_n$. Again we denote by x_w the flag

$$x_w^1 \subset \cdots \subset x_w^n$$

built as in Section 2. Then the constructions of Section 2 can be immediately translated into the following property.

THEOREM 6. *For all $w \in \text{Sym}_n$ and x a maximal chain in L ,*

$$\pi(w, x) = \pi(w, x_w).$$

PROOF. Since the property which we need to verify involves only one fixed permutation w , we can assume, after appropriately renumbering the atoms $1, \dots, n$, that the permutation w is the identity permutation, $w = 1$, and the flag w equals the chain

$$e = \{1 < 1 \vee 2 < \cdots < 1 \vee \cdots \vee n\}.$$

So we need to check the identity

$$\pi(e, x) = \pi(e, x_e).$$

By Eqn. (2),

$$\pi(e, x)(j) = i \quad \text{if} \quad \begin{cases} 1 \vee \cdots \vee (i-1) \vee x_k = 1 \vee \cdots \vee i \vee x_k & \text{for } k \geq j \\ 1 \vee \cdots \vee (i-1) \vee x_k < 1 \vee \cdots \vee i \vee x_k & \text{for } k < j. \end{cases}$$

On the other hand, Eqn. (1) defines the set x_e^k as the set of those i for which

$$1 \vee 2 \vee \cdots \vee (i-1) \vee x_k = 1 \vee 2 \vee \cdots \vee i \vee x_k.$$

We immediately see from these two equations that $\pi(e, x)([k]) \subseteq x_e^k$. Since $|x_e^k| = k$, $\pi(e, x) = x_e^k$. But this means exactly that $\pi(e, x) = \pi(e, x_e)$. \square

COROLLARY 7. *For all w in Sym_n and x a maximal chain in L ,*

$$\rho_w(x) = x_w.$$

Hence

$$\mu_x(w) = x_w,$$

and the flag matroid corresponding to the matroid map μ_x is the collection of all flags of subsets $\{x_w : w \in \text{Sym}_n\}$.

This theorem shows that an arbitrary matroid map may be represented as a chamber in the chamber complex of a semimodular lattice. In another paper [4] we will show that all matroid maps on Sym_n for a given n may be simultaneously represented in another chamber complex with a somewhat weaker version of a Sym_n metric.

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