

# Lagrangian Pairs and Lagrangian Orthogonal Matroids

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**Abstract.** Lagrangian orthogonal matroids are the best-behaved Coxeter matroids, except for ordinary matroids, and indeed they include ordinary matroids as a special case. Represented Lagrangian orthogonal matroids arise from  $n$ -dimensional totally orthogonal subspaces of  $2n$ -dimensional orthogonal space, and if we take the unique pair of such subspaces which contain a given  $n - 1$ -dimensional totally isotropic subspace, we get the prototype of a Lagrangian pair of Lagrangian orthogonal matroids. In this paper we define Lagrangian pairs, and give a number of characterizations of them and prove

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several properties of them. We also use them to prove a new characterization, within ordinary matroid theory, of an elementary quotient of ordinary matroids.

**Introduction.** Represented Coxeter matroids of types  $C_n$  and  $D_n$ , that is, symplectic and orthogonal matroids arising from totally isotropic subspaces of symplectic or (even-dimensional) orthogonal spaces, may also be represented in buildings of type  $C_n$  and  $D_n$ , respectively (see [4, Chapter 7]). Indeed, the particular buildings involved are those arising from the flags or oriflammes, respectively, of totally isotropic subspaces. There are also buildings of type  $B_n$  arising from flags of totally isotropic subspaces in odd-dimensional orthogonal space. Coxeter matroids of type  $B_n$  are the same as those of type  $C_n$  (since they depend only upon the directions of roots, not the length of roots—see the Gelfand-Serganova Theorem below). However, buildings of type  $B_n$  are distinct from those of the other types. Thus the question arises whether there are such things as  $B_n$ -representable matroids, that is, those representable in odd dimensional orthogonal space, and presumably therefore in such buildings. We answer this question in the affirmative in [2]. The matroids so representable turn out to be a special case of symplectic (flag) matroids, those whose top component, or Lagrangian matroid, is a union of two Lagrangian orthogonal matroids. These two matroids are called a Lagrangian pair, and they are the combinatorial manifestation of the “fork” at the top of an oriflamme (or of the fork at the end of the Coxeter diagram of  $D_n$ ). Thus Lagrangian pairs are a very natural subject of investigation.

## 1 Symplectic and Orthogonal Matroids

Let

$$[n] = \{1, 2, \dots, n\} \text{ and } [n]^* = \{1^*, 2^*, \dots, n^*\}.$$

Define the map  $*$  :  $[n] \rightarrow [n]^*$  by  $i \mapsto i^*$  and the map  $*$  :  $[n]^* \rightarrow [n]$  by  $i^* \mapsto i$ . In other words, we are defining  $i^{**} = i$ . Then  $*$  is an involutive permutation of the set  $[n] \cup [n]^*$ .

We say that a subset  $K \subset [n] \cup [n]^*$  is *admissible* if and only if  $K \cap K^* = \emptyset$ .

A linear ordering  $\prec$  of  $[n] \cup [n]^*$  is called a  $C_n$ -*admissible ordering* if  $i \prec j$  implies that  $j^* \prec i^*$  for all  $i, j \in [n] \cup [n]^*$ . Equivalently, an ordering  $\prec$  on  $[n] \cup [n]^*$  is  $C_n$ -admissible if and only if, when the  $2n$  elements are listed from largest to smallest, the first  $n$  elements listed form an admissible set, and the last  $n$  elements listed are the stars of the first  $n$  elements listed, but are

listed in reverse order. A  $D_n$ -admissible ordering of  $[n] \cup [n]^*$  is similar to a  $C_n$ -admissible ordering, except that the middle two elements (i.e., the  $n$ -th and  $n + 1$ -st elements in the above listing) are now incomparable.

Denote by  $J_k$  the collection of all admissible  $k$ -subsets in  $J$ , for some  $k \leq n$ . If  $\prec$  is  $C_n$  or  $D_n$ -admissible ordering on  $[n] \cup [n]^*$ , it induces the partial ordering (which we denote by the same symbol  $\prec$ ) on  $J_k$ : if  $A, B \in J_k$  and

$$A = \{a_1 \prec a_2 \prec \cdots \prec a_k\} \quad \text{and} \quad B = \{b_1 \prec b_2 \prec \cdots \prec b_k\},$$

we set  $A \prec B$  if

$$a_1 \prec b_1, a_2 \prec b_2, \dots, a_k \prec b_k.$$

This partial ordering is called the Gale ordering on  $J_k$  induced by  $\prec$ . Given an arbitrary partial order  $\leq$  on a set  $T$ , we can also induce a partial ordering  $\leq$  on  $T \times T$  by saying  $(s, t) \leq (u, v)$  whenever  $s \leq u$  and  $t \leq v$ . We can likewise induce a partial order on unordered pairs of elements of  $T$  by setting  $\{s, t\} \leq \{u, v\}$  whenever  $(s, t) \leq (u, v)$  or  $(s, t) \leq (v, u)$ .

Now let  $\mathcal{B} \subseteq J_k$  be a collection of admissible  $k$ -element subsets of the set  $J$ . We say that  $M = (*, \mathcal{B})$  is a *symplectic matroid* if it satisfies the following *Maximality Property*:

*for every  $C_n$ -admissible order  $\prec$  on  $J$ , the collection  $\mathcal{B}$  contains a unique maximal member, i.e. a subset  $A \in \mathcal{B}$  such that  $B \prec A$  (in the Gale order induced by  $\prec$ ), for all  $B \in \mathcal{B}$ .*

The collection  $\mathcal{B}$  is called the *collection of bases* of the symplectic matroid  $M$ , its elements are called *bases* of  $M$ , and the cardinality  $k$  of the bases is the *rank* of  $M$ . An *orthogonal matroid* is defined similarly using  $D_n$ -admissible orderings. Ordinary matroids on  $[n]$  can be defined in similar fashion, using  $A_n$ -admissible orderings, which are arbitrary linear orderings on  $[n]$ ; indeed, this definition is essentially the well-known greedy algorithm characterisation of matroid theory. A *Lagrangian matroid* (resp. *Lagrangian orthogonal matroid*) is a symplectic matroid (resp. orthogonal matroid) of rank  $n$ , the maximum possible.

A useful characterization of symplectic and orthogonal matroids is given by the Gelfand-Serganova Theorem (see [4]). For  $B \in \mathcal{B}$ , define a point in a real vector space spanned by  $\{\epsilon_i : i \in [n]\}$  by

$$\delta_B = \sum_{j \in B} \epsilon_j,$$

where  $\epsilon_{i^*}$  is defined to be  $-\epsilon_i$ . Then  $\Delta_{\mathcal{B}}$  is defined to be the convex hull of the  $\delta_B$  for  $B \in \mathcal{B}$ . We define roots for  $D_n$  to be all vectors of the form  $\epsilon_j - \epsilon_k$

for  $j, k \in [n] \cup [n]^*$ . Roots for  $C_n$  are the same together with all vectors of the form  $2\epsilon_j$  for  $j \in [n] \cup [n]^*$ . Then the *Gelfand-Serganova Theorem* says that if  $\mathcal{B} \subseteq J_k$ , then  $\mathcal{B}$  is a symplectic (resp. orthogonal) matroid if and only if  $\Delta_{\mathcal{B}}$  has all of its edges (i.e., one-dimensional faces) parallel to roots for  $C_n$  (resp.  $D_n$ ).

In the case of a Lagrangian orthogonal matroid, if  $\delta_A$  and  $\delta_B$  are adjacent vertices in  $\Delta$ , then the edge between them is parallel to a root  $\epsilon_j - \epsilon_k$ . Since  $A$  and  $B$  are admissible  $n$ -sets, each must have either  $j$  or  $j^*$  as an element, and likewise  $k$  or  $k^*$ . It follows that  $B = (j, k)(j^*, k^*)A$ , regarded as a permutation (in cycle notation) acting on  $A$ . Whether  $j$  and  $k$  are in  $[n]$  or  $[n]^*$ , it follows that  $B$  has the same parity as  $A$  in terms of number of starred elements. Consequently, all bases of a Lagrangian orthogonal matroid have the same parity (Note: the converse is false). The parity condition is not true for non-Lagrangian orthogonal matroids; for example  $(1, 2^*)(1^*, 2)\{1\} = \{2^*\}$ .

Another characterization of Lagrangian orthogonal matroids comes from cosets in the group  $D_n$ . As a permutation group on  $[n] \cup [n]^*$ ,  $D_n$  is generated by the involutions

$$\begin{aligned} s_1 &= (1, 2)(1^*, 2^*) \\ s_2 &= (2, 3)(2^*, 3^*) \\ &\vdots \\ s_{n-1} &= (n-1, n)((n-1)^*, n^*) \\ s_n &= (n-1, n^*)((n-1)^*, n) \end{aligned}$$

in cycle notation. Consider the two maximal parabolic subgroups  $P^n$  and  $P^{n-1}$ , generated by  $\{s_1, s_2, \dots, s_{n-1}\}$  and  $\{s_1, s_2, \dots, s_{n-2}, s_n\}$ , respectively. Notice that  $P^n$  is the stabilizer in  $D_n$  of the admissible  $n$ -set  $[n]$ , and hence the left cosets of  $P^n$  in  $D_n$  correspond to the orbit of  $[n]$ , namely, all admissible  $n$ -sets of even parity. Similarly,  $P^{n-1}$  is the stabilizer of  $\{1, 2, \dots, n-1, n^*\}$ , and its left cosets correspond to all admissible  $n$ -sets of odd parity. Furthermore, in similar fashion, left cosets of  $P^n \cap P^{n-1}$  can be shown to correspond to admissible  $(n-1)$ -sets. Letting elements of  $D_n$  act on  $D_n$ -admissible orderings in the obvious way, we find that  $D_n$  corresponds bijectively to the set of all  $D_n$ -admissible orderings. Now we can characterize Lagrangian orthogonal matroids  $M$  of even parity as maps from  $D_n$  to  $D_n/P^n$  (the set of all left cosets of  $P^n$  in  $D_n$ ). An element  $\sigma$  of  $D_n$  is sent to the coset corresponding to the maximal basis of  $M$  given by the Maximality Property for the admissible order corresponding to  $\sigma$ . This *matroid map*  $\mu : D_n \rightarrow D_n/P^n$  is actually very natural, and characterises the Lagrangian orthogonal matroids of even

parity, since the Maximality Property can be entirely rephrased in terms of the cosets using Bruhat order, see [4]. Likewise, Lagrangian orthogonal matroids of odd parity correspond to matroid maps  $\mu : D_n \rightarrow D_n/P^{n-1}$ .

One more very useful characterization of Lagrangian orthogonal matroids is the Strong Exchange Property [1]. A collection  $\mathcal{B} \subseteq J_n$  is the collection of bases of a Lagrangian orthogonal matroid if and only if:

For every  $A, B \in \mathcal{B}$  and  $a \in A \setminus B$ , there exists  $b \in B \setminus A$  with  $b \neq a^*$ , such that both  $A \triangle \{a, b, a^*, b^*\}$  and  $B \triangle \{a, b, a^*, b^*\}$  are members of  $\mathcal{B}$ .

## 2 Characterisations of Lagrangian Pairs

Consider an admissible set of size  $n - 1$ . Such a set can be completed to an admissible set of size  $n$  in exactly two ways, by appending either  $i$  or  $i^*$  for some  $i$ . The two resulting sets are called a *Lagrangian pair* of sets, and are characterised by the fact that their symmetric difference is exactly  $\{i, i^*\}$ .

Consider now two Lagrangian orthogonal matroids  $M_1, M_2$  of rank  $n$  and of opposite parity. We say that they form a Lagrangian pair (of Lagrangian orthogonal matroids) if they satisfy:

For every admissible ordering, the maximal bases of  $M_1$  and  $M_2$  under the ordering are a Lagrangian pair of sets.

We say that a pair of Lagrangian subspaces of orthogonal  $2n$ -space form a Lagrangian pair of subspaces if their intersection is of dimension  $n - 1$ . The following lemma is well-known.

**Lemma 1** *A totally isotropic subspace of dimension  $n - 1$  in orthogonal  $2n$ -space is contained in exactly two Lagrangian subspaces (which are a Lagrangian pair).*

**Theorem 2** *A Lagrangian pair of subspaces represent a Lagrangian pair of orthogonal matroids.*

**Proof** Let  $U_1$  and  $U_2$  be a Lagrangian pair of subspaces of a  $2n$ -dimensional orthogonal space. Let a  $D_n$ -admissible ordering be given. The totally isotropic subspace  $U_1 \cap U_2$  can be represented by an  $(n-1) \times 2n$  matrix  $C$ , with columns indexed by elements of  $[n] \cup [n]^*$ , see [4, Chapter 3]. Reorder the columns so that they are in the given ordering, where the two columns indexed by the two unrelated elements may be put in either order. Let  $C'$  be the reduced

row echelon form of  $C$ . Then the pivot columns of  $C'$  are those indexed by the maximal basis  $B$  of the rank  $n - 1$  orthogonal matroid represented by  $C$ , and, in particular, the pivot columns must be indexed by an admissible set. Either  $U_1$  or  $U_2$  may be similarly represented by adding one row to  $C$  and again row reducing. However, by elementary linear algebra, the original  $n - 1$  pivot columns remain pivot columns, with one additional pivot column being added in each case. Thus the maximal bases of the Lagrangian orthogonal matroids represented by  $U_1$  and  $U_2$  are both admissible  $n$ -sets containing  $B$ , hence are a Lagrangian pair of sets. Thus these two matroids are a Lagrangian pair.  $\diamond$

Let  $M_1, M_2$  be Lagrangian orthogonal matroids on  $[n] \cup [n]^*$  of opposite parity,  $\mathcal{B}_1, \mathcal{B}_2$  their collections of bases, and

$$\mu_1 : W \rightarrow D_n/P^n, \quad \mu_2 : W \rightarrow D_n/P^{n-1}$$

the corresponding matroid maps. Let

$$\begin{aligned} \mathcal{B}_1 + (n + 1) &= \{ B \cup \{ n + 1 \} \mid B \in \mathcal{B}_1 \}, \\ \mathcal{B}_2 + (n + 1)^* &= \{ B \cup \{ (n + 1)^* \} \mid B \in \mathcal{B}_2 \}, \\ \mathcal{B}_3 &= (\mathcal{B}_1 + (n + 1)) \cup (\mathcal{B}_2 + (n + 1)^*), \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}_4 &= \{ A \mid A \subseteq [n] \cup [n]^*, |A| = n - 1, \\ &\quad \text{and there exist } B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2 \\ &\quad \text{such that } A = B_1 \cap B_2 \}. \end{aligned}$$

We call  $\mathcal{B}_3$  the (collection of bases of the) *exploded sum* of  $M_1$  and  $M_2$ , see Figure 1.

**Theorem 3** *The following are equivalent:*

- (1)  $M_1$  and  $M_2$  are a Lagrangian pair,
- (2) for all  $w \in W$ ,  $\mu_1(w) \cap \mu_2(w) \neq \emptyset$ ,
- (3)  $\mathcal{B}_3$  is the collection of bases of a Lagrangian orthogonal matroid,
- (4)  $\mathcal{B}_4$  is the collection of bases of a Lagrangian orthogonal matroid, and for each  $B \in \mathcal{B}_1 \cup \mathcal{B}_2$  there exists  $X \in \mathcal{B}_4$  with  $X \subset B$ .

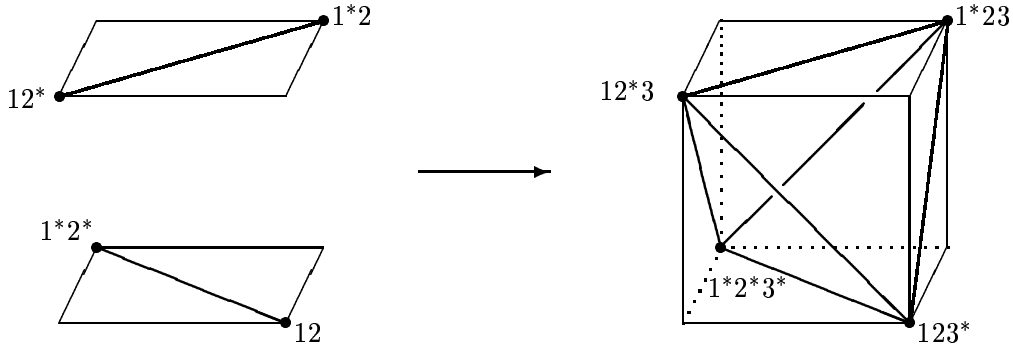


Figure 1: *The exploded sum of a Lagrangian pair of Lagrangian orthogonal matroids.*

- (5) *for all  $w \in W$ , there exists a unique unordered pair  $\{B_1, B_2\}$ , with  $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$  and  $|B_1 \cap B_2| = n - 1$ , such that for all  $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$ , we have  $\{A_1, A_2\} \leq^w \{B_1, B_2\}$ ,*
- (6) *for all  $w \in W$ , there exists a unique ordered pair  $(B_1, B_2)$ , with  $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$  and  $|B_1 \cap B_2| = n - 1$ , such that for all  $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$ , we have  $(A_1, A_2) \leq^w (B_1, B_2)$ .*

**Proof** The equivalence (4)  $\Leftrightarrow$  (5) is proved in [6]. Furthermore, (1)  $\Leftrightarrow$  (6), (6)  $\Rightarrow$  (5), and (5)  $\Rightarrow$  (2) are immediate. We will now prove (1)  $\Rightarrow$  (3), (3)  $\Rightarrow$  (1), and (2)  $\Rightarrow$  (1).

Assume that  $M_1$  and  $M_2$  are a Lagrangian pair, and choose a  $D_{n+1}$ -admissible order on  $[n + 1] \cup [n + 1]^*$ . Restrict this ordering to  $[n] \cup [n]^*$ . Notice that now this may be either a  $C_n$  or  $D_n$ -admissible order. Let  $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$  be the maximal bases in this restricted order. If the restricted order is a  $C_n$ -admissible order, it can be changed to a  $D_n$ -admissible order by deleting the relation between the pair of elements in the middle, and the maximal bases remain unchanged. **This is because we know that  $M_1$  and  $M_2$  each has a unique maximal basis in the  $D_n$ -admissible order, and changing the  $D_n$ -admissible order to the  $C_n$ -admissible order cannot change the maximal bases.** Hence  $B_1 \Delta B_2 = \{i, i^*\}$  for some  $i$ . Clearly  $B'_1 = B_1 \cup \{n + 1\}$  and  $B'_2 = B_2 \cup \{(n + 1)^*\}$  are the only two candidates for maximal members of  $\mathcal{B}_3$ . But  $B'_1 \Delta B'_2 = \{i, i^*, n + 1, (n + 1)^*\}$ , and in all possible  $D_{n+1}$ -admissible orders, **each 2-element admissible subset**

of  $B'_1 \triangle B'_2$  is related to its complement, hence  $B'_1$  and  $B'_2$  are related. Thus  $\mathcal{B}_3$  has a unique maximal member, proving (3).

Now assume (3), and let a  $D_n$ -admissible order be given. Let  $B_1$  (resp.  $B_2$ ) be the maximal basis in  $\mathcal{B}_1$  (resp.  $\mathcal{B}_2$ ). Extend the given order arbitrarily to a  $D_{n+1}$ -admissible order, and let  $f$  be a linear functional compatible with the extended order. By this we mean that  $f$  is a linear functional on the real vector space spanned by the basis  $\{\epsilon_1, \epsilon_2, \dots, \epsilon_{n+1}\}$ , with  $\epsilon_i = -\epsilon_{i^*}$  by definition, such that  $i \prec j$  in the extended order implies  $f(i) < f(j)$ , for all  $i, j \in [n+1] \cup [n+1]^*$ . Now  $B'_1 = B_1 \cup \{n+1\}$  (resp.  $B'_2 = B_2 \cup \{(n+1)^*\}$ ) is clearly the maximal basis in

$$\mathcal{B}'_1 = \{B \cup \{n+1\} \mid B \in \mathcal{B}_1\}$$

(resp.  $\mathcal{B}'_2 = \{B \cup \{(n+1)^*\} \mid B \in \mathcal{B}_2\}$ ). Thus  $\delta_{B'_1}$  (resp.  $\delta_{B'_2}$ ) is the unique maximal vertex of  $\Delta_{\mathcal{B}'_1}$  (resp.  $\Delta_{\mathcal{B}'_2}$ ) under  $f$ . Since  $\Delta_{\mathcal{B}'_1}$  and  $\Delta_{\mathcal{B}'_2}$  lie in parallel hyperplanes, it is easy to see that  $\delta_{B'_1} \delta_{B'_2}$  must be an edge of  $\Delta_{\mathcal{B}_3}$ . By the Gelfand-Serganova Theorem,  $B'_1$  and  $B'_2$  must be related by an exchange of the form  $(n+1, (n+1)^*)(i, i^*)$  (in cycle notation), for some  $i \leq n$ . It follows that  $B_1$  and  $B_2$  must be related by the exchange  $(i, i^*)$ , proving that  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are a Lagrangian pair.

Finally, assume (2). Thus  $\mu_1(w) = aP^n$  and  $\mu_2(w) = bP^{n-1}$  with  $aP^n \cap bP^{n-1} \neq \emptyset$ . But this means  $aP^n \cap bP^{n-1} = c(P^n \cap P^{n-1}) = cP^{n,n-1}$ . It follows that there is an admissible  $(n-1)$ -set contained in both  $B_1 = \max^w \mathcal{B}_1$  and  $B_2 = \max^w \mathcal{B}_2$ , and thus that  $B_1$  and  $B_2$  differ by an exchange of the form  $(i, i^*)$  for some  $i$ , showing that  $M_1$  and  $M_2$  are a Lagrangian pair.  $\diamond$

Condition (2) in the previous theorem amounts to saying that the two Lagrangian orthogonal matroids are *concordant*, see [3].

### 3 Further results on Lagrangian pairs

We now need to recall the concept of quotient (or, essentially, strong map) of ordinary matroids. If  $M_1$  and  $M_2$  are matroids on the same set  $[n]$ , then we say that  $M_2$  is a quotient of  $M_1$  if every circuit of  $M_1$  is a union of circuits of  $M_2$ . As is shown in [4, Chapter 1],  $M_2$  is a quotient of  $M_1$  if and only if, for every linear ordering of  $[n]$ , the maximal basis of  $M_2$  is a subset of the maximal basis of  $M_1$ . To relate ordinary matroids to Lagrangian orthogonal matroids, we need a mapping  $\Phi$  defined as follows: For  $B \subseteq [n]$ , let  $\Phi(B) = B \cup ([n] \setminus B)^*$ . Then if  $\mathcal{B}$  is the collection of bases of a matroid,  $\Phi(\mathcal{B}) = \{\Phi(B) \mid B \in \mathcal{B}\}$  is a Lagrangian orthogonal matroid, as proved in

[4, Chapter 3]. An elementary quotient is one in which the two matroids differ by 1 in rank.

**Theorem 4** *Let  $M_1$  and  $M_2$  be ordinary matroids on  $[n]$ , of ranks  $k$  and  $k - 1$  respectively. Then  $M_2$  is a quotient (thus an elementary quotient) of  $M_1$  if and only if  $\Phi(M_1)$  and  $\Phi(M_2)$  are a Lagrangian pair.*

**Proof** Let  $\prec$  be a  $D_n$ -admissible ordering of  $[n] \cup [n]^*$ , and let  $\leq$  denote the restriction of this ordering to  $[n]$ , which must be a linear order. First we claim that if  $A$  and  $B$  are bases of the same matroid  $M$ , then  $A \leq B$  implies  $\Phi(A) \prec \Phi(B)$ . Indeed, if  $A \leq B$ , then  $[n] \setminus A \geq [n] \setminus B$ , and hence  $([n] \setminus A)^* \leq ([n] \setminus B)^*$ , so  $\Phi(A) \prec \Phi(B)$ . It follows that  $B$  is the maximal basis of  $M$  if and only if  $\Phi(B)$  is the maximal basis of  $\Phi(M)$ .

Now let  $B_1$  and  $B_2$  be the maximal bases of  $M_1$  and  $M_2$ , resp. If  $M_2$  is a quotient of  $M_1$ , then  $B_2 \subseteq B_1$ , say  $B_2 = B_1 \setminus \{i\}$ . Then  $\Phi(B_1)$  and  $\Phi(B_2)$  differ by the exchange  $(i, i^*)$ . But we have just seen that these are the maximal bases of  $\Phi(M_1)$  and  $\Phi(M_2)$ , resp., showing that these two Lagrangian orthogonal matroids are a Lagrangian pair.

Conversely, suppose that  $M_2$  is not a quotient of  $M_1$ . Then there exists a linear ordering on  $[n]$  such the maximal bases  $B_1$  and  $B_2$  of  $M_1$  and  $M_2$  (resp.) are not related by containment. Extend this ordering to a  $D_n$ -admissible ordering on  $[n] \cup [n]^*$ . Then the maximal bases  $\Phi(B_1)$  and  $\Phi(B_2)$  of  $\Phi(M_1)$  and  $\Phi(M_2)$  (resp.) are not related by an exchange of the form  $(i, i^*)$ . Thus  $\Phi(M_1)$  and  $\Phi(M_2)$  are not a Lagrangian pair.  $\diamond$

As a corollary, we get a new characterization of elementary quotients.

**Corollary 5** *Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be the collection of bases of two matroids  $M_1$  and  $M_2$  of ranks  $k$  and  $k - 1$ , respectively. Then  $M_2$  is an elementary quotient of  $M_1$  if and only if for every  $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$ , and  $i \in B_1 \Delta B_2$ , either*

- (1)  $B_1 \Delta \{i\} \in \mathcal{B}_2, B_2 \Delta \{i\} \in \mathcal{B}_1$ , or
- (2) there exists  $j \in B_1 \Delta B_2$  such that

$$B_1 \Delta \{i, j\} \in \mathcal{B}_1 \quad \text{and} \quad B_2 \Delta \{i, j\} \in \mathcal{B}_2,$$

and furthermore, for every  $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$  there exists  $i \in B_1 \Delta B_2$  such that case (1) holds.

**Proof** We have that  $M_2$  is an elementary quotient of  $M_1$  if and only if  $\Phi(M_1)$  and  $\Phi(M_2)$  are a Lagrangian pair if and only if their exploded sum is a Lagrangian orthogonal matroid, which is true if and only if their exploded sum satisfies the Strong Exchange Property. The corollary follows immediately from translating what the Strong Exchange Property says in terms of  $\mathcal{B}_1$  and  $\mathcal{B}_2$ .  $\diamond$

**Theorem 6** *Let  $\mathcal{B}$  be a Lagrangian orthogonal matroid on  $[n] \cup [n]^*$ , and  $i \in [n]$ . Define*

$$\begin{aligned}\mathcal{B}_1 &= \{ B \setminus \{i\} \mid i \in B, B \in \mathcal{B} \}, \\ \mathcal{B}_2 &= \{ B \setminus \{i^*\} \mid i^* \in B, B \in \mathcal{B} \}.\end{aligned}$$

**If  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are both non-empty, then  $\mathcal{B}_1, \mathcal{B}_2$  are a Lagrangian pair of Lagrangian orthogonal matroids on  $([n] \cup [n]^*) \setminus \{i, i^*\}$ .**

**Proof** From the Gelfand-serganova Theorem, it is easy to see that  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are Lagrangian orthogonal matroids, given that they are non-empty. We see that  $\mathcal{B}$  is just the exploded sum of  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , after appropriate relabelling, so the desired result follows from Theorem 3.  $\diamond$

**Theorem 7** *Let  $\mathcal{B}_1, \mathcal{B}_2$  be the collections of bases of a Lagrangian pair of Lagrangian orthogonal matroids. Then  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$  is the collection of bases of a Lagrangian (symplectic) matroid.*

**Proof** Let  $\mathcal{B}_1, \mathcal{B}_2$  be a Lagrangian pair, and let a  $C_n$ -admissible order be given. Then the mximal bases  $B_1$  and  $B_2$  (resp.), which must be the same as the maximal bases in the corresponding  $D_n$ -admissible order, are related by an exchange  $\{i, i^*\}$ , and are thus related in the  $C_n$ -admissible order. Thus the larger of the two is the unique maximal basis in  $\mathcal{B}$ .  $\diamond$

Since the union of a Lagrangian pair is a symplectic matroid, it is natural to wonder whether, given Lagrangian orthogonal matroids of the same rank and opposite parity whose union is a symplectic matroid, they are necessarily a Lagrangian pair. Figure 2 shows that the answer is no. Indeed, since 123 is the only basis of the matroid of even parity, any admissible order which makes 1\*2\*3\* the maximal basis of the other matroid violates the definition of Lagrangian pair.

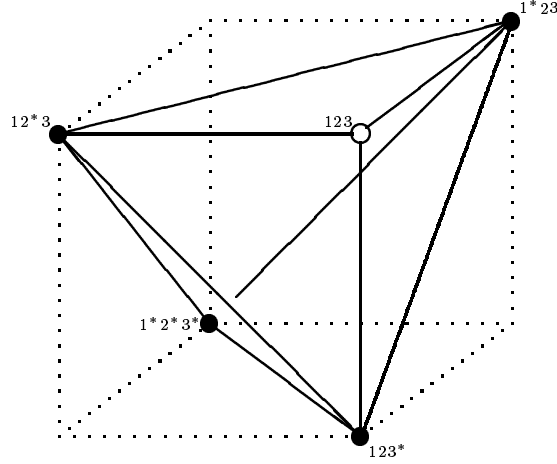


Figure 2: A Lagrangian symplectic matroid which is not the union of a Lagrangian pair.

**Theorem 8** Let  $\mathcal{B}_1$  be the collection of bases of a Lagrangian orthogonal matroid, and  $\mathcal{B}_2 = (i, i^*)\mathcal{B}_1$ , where  $(i, i^*)$  is a transposition, to be thought of as a permutation acting elementwise on the members of  $\mathcal{B}_1$ . Then  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are a Lagrangian pair.

**Proof** By Theorem 3, it suffices to prove that the exploded sum,

$$\mathcal{B}_3 = (\mathcal{B}_1 + (n+1)) \cup (\mathcal{B}_2 + (n+1)^*)$$

is a Lagrangian orthogonal matroid. So we wish to show that  $\mathcal{B}_3$  satisfies the Strong Exchange Property. Let  $A, B \in \mathcal{B}_3$  and  $a \in A \setminus B$ . We need to show that there exists  $b \in (A \triangle B) \setminus \{a, a^*\}$  so that  $A \triangle \{a, a^*, b, b^*\}$  and  $B \triangle \{a, a^*, b, b^*\}$  are both in  $\mathcal{B}_3$ . If both  $A$  and  $B$  are in  $\mathcal{B}_1 + (n+1)$  or else both in  $\mathcal{B}_2 + (n+1)^*$ , then we are done, for  $a \neq (n+1), (n+1)^*$ , and Strong Exchange holds in  $\mathcal{B}_1$  and in  $\mathcal{B}_2$ . Thus we may assume that  $A \in \mathcal{B}_1 + (n+1)$  and  $B \in \mathcal{B}_2 + (n+1)^*$ .

Case 1. Suppose  $a = i$  or  $i^*$ . Then we may choose  $b = n+1$ , and we are done.

Case 2. Suppose  $a = n+1$ . If  $i \in A \triangle B$ , then we choose  $b = i$  and we are done. Thus we suppose, without loss of generality, that  $i \in A \cap B$ . Let  $B' = B \triangle \{i, i^*, n+1, (n+1)^*\}$ . Then  $B' \in \mathcal{B}_1 + (n+1)$ , and  $i \in A \triangle B'$ .

By Strong Exchange on  $\mathcal{B}_1$ , we have  $j \in [n] \cup [n]^*$  so that

$$A' := A \Delta \{i, i^*, j, j^*\}$$

and

$$B' \Delta \{i, i^*, j, j^*\} = B \Delta \{j, j^*, n+1, (n+1)^*\}$$

are both in  $\mathcal{B}_1 + (n+1)$ . Thus

$$\begin{aligned} A \Delta \{j, j^*, n+1, (n+1)^*\} &= A' \Delta \{i, i^*, n+1, (n+1)^*\} \\ &\in \mathcal{B}_2 + (n+1)^*. \end{aligned}$$

Thus  $b = j$  gives the desired Strong Exchange.

Case 3. We are left with  $a \neq i, i^*, n+1, (n+1)^*$ . Let

$$B' = B \Delta \{i, i^*, n+1, (n+1)^*\} \in \mathcal{B}_1 + (n+1).$$

Since  $a \in A \Delta B'$ , by Strong Exchange in  $\mathcal{B}_1$ , there exists  $j \in A \Delta B'$ ,  $j \neq a, a^*, n+1, (n+1)^*$ , so that

$$A' := A \Delta \{a, a^*, j, j^*\} \in \mathcal{B}_1 + (n+1)$$

and  $B' \Delta \{a, a^*, j, j^*\} \in \mathcal{B}_1 + (n+1)$ . Then

$$\begin{aligned} B \Delta \{a, a^*, j, j^*\} &= B' \Delta \{i, i^*, n+1, (n+1)^*\} \Delta \{a, a^*, j, j^*\} \\ &\in \mathcal{B}_2 + (n+1)^*, \end{aligned}$$

regardless of whether  $j = i$  or  $i^*$ , or not. Thus  $b = j$  again gives the desired Strong Exchange.  $\diamond$

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