Spring 2008 Calculus 3 Problems

Challenge problems are indicated with [C].

1 Fix $a, b > 0$. Show that πab is the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

SKETCH OF SOLUTION. Area $= \int_{-a}^{a} 2b\sqrt{1-(x/a)^2}\,dx$. Then use substitution $x/a = \sin\theta$.

2 Compute $\displaystyle\int_0^\infty xe^{-x^2}\,dx$.

SKETCH OF SOLUTION. Answer 1/2. Improper integral: use substitution $u = -x^2$.

3 Find the maximum and minimum values of

$$f(x) = -x^2 - 4x + 50$$

in the interval $-4 \le x \le 4$.

SKETCH OF SOLUTION. $f'(x) = -2x - 4 = 0$ gives critical point $x = -2$, and $f(-2) = 54$. Check endpoints: $f(4) = 18$, $f(-4) = 50$. Max 54, Min 18.

4 Find the (minimum) distance from the point $(1, 2)$ to the line $x - y = 4$.

SKETCH OF SOLUTION. Answer $\frac{5\sqrt{2}}{2}$. Minimize $(1-x)^2 + (2-y)^2 = (1-x)^2 + (6-x)^2 = 2x^2 - 14x + 37$.

5 Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

SKETCH OF SOLUTION. $V = \displaystyle\int_0^1 \pi(\sqrt{x})^2\,dx = \frac{\pi}{2}$.

6 When a ball is located a distance x meters from the origin a force of x^2 newtons acts on it. Find the work done in moving the ball from $x = 2$ meters to $x = 3$ meters.

SKETCH OF SOLUTION. $W = \int_a^b f(x)\,dx = \int_2^3 x^2\,dx = \frac{19}{3}$ Joules.

7 Consider the curve described parametrically by $x = 1-t$ and $y = 2-4t+t^2$. Eliminate the parameter and sketch the curve.

SKETCH OF SOLUTION. $y = 2-4t+t^2 = 2-4(1-x)+(1-x)^2 = x^2+2x-1$. Parabola oriented right to left.

8 Sketch the cycloid $x = 2(t - \sin t), y = 2(1 - \cos t)$, and determine the points where the tangent line is horizontal.

SKETCH OF SOLUTION. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\sin t}{1-\cos t}$. Tangent line is horizontal when $\sin t = 0$ and $1 - \cos t \neq 0$, which implies that $t = (2n-1)\pi, n$ an integer. The points are $(x, y) = (2(2n-1)\pi, 4)$.

9 Find the area of the region $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} \leq 1$.

SKETCH OF SOLUTION. $A = 4\displaystyle\int_0^1 (1-\sqrt{x})^2\, dx = 2/3$.

10 Sketch the polar curve $r = 1 + \cos\theta$.

SKETCH OF SOLUTION. Cardioid

11 (§12.1) Determine the center and radius of the sphere given by the equation $x^2 + y^2 + z^2 = -2x + 4y$.

SKETCH OF SOLUTION. $(x+1)^2 + (y-2)^2 + (z-0)^2 = 5$ with center (-1,2,0) and radius $\sqrt{5}$.

12 (§12.1) Find the minimum distance from the point $(3, 7, -5)$ to the plane $z = 3$.

SKETCH OF SOLUTION. $d = \sqrt{(x-3)^2 + (y-7)^2 + (3-(-5))^2}$ is minimized when $x = 3, y = 7$. The minimum distance is 8.

13 (§12.1) Find an equation of the set of all points equidistant from the points $(1, 2, 0)$ and $(-1, 0, 0)$. Describe the set.

SKETCH OF SOLUTION. $x + y = 1$, which is the equation of a plane.

14 (§12.2) Let $\vec{a} = \langle -2, 3\rangle$ and $\vec{b} = \langle 1, 2\rangle$. Compute $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$. Sketch.

SKETCH OF SOLUTION. $\vec{a} + \vec{b} = \langle -1, 5\rangle$ and $\vec{a} - \vec{b} = \langle -3, 1\rangle$.

15 (§12.2) Two tug boats are pulling a barge. Choosing the desired line of travel as the x-axis, one tug is pulling with a force of magnitude 20 and angle of $\frac{\pi}{4}$ and the second with a magnitude of 15 and angle of $-\frac{\pi}{6}$. Find the magnitude and direction of the resulting force.

SKETCH OF SOLUTION. $\vec{F_1} = [20\cos(45°)]\ \vec{\imath} + [20\sin(45°)]\ \vec{\jmath}$ and $\vec{F_2} = [15\cos(-30°)]\ \vec{\imath} + [15\sin(-30°)]\ \vec{\jmath}$. $\vec{F_1} + \vec{F_2} = \langle 10\sqrt{2} + 15\sqrt{3}/2, 10\sqrt{2} - 15/2\rangle \approx \langle 27.13, 6.64\rangle$. $|\vec{F_1} + \vec{F_2}| \approx 27.9$ and direction $\theta \approx 14°$.

16 (§12.2, 12.3) (a) Find the unit vector $\vec{u}$ in the direction $\vec{v} = 6\vec{\imath} + 8\vec{\jmath}$.

(b) Find the projection of $\vec{w} = 2\vec{\imath} + \vec{\jmath}$ on $\vec{u}$. (c) Find $\text{proj}_{\vec{v}}\vec{w}$.

SKETCH OF SOLUTION. (a) $\vec{u} = \langle 3/5, 4/5\rangle$ (b) $\text{proj}_{\vec{u}}\vec{w} = \langle 6/5, 8/5\rangle$ (c) $\text{proj}_{\vec{v}}\vec{w} = \langle 6/5, 8/5\rangle$, the same!

17 (§12.3) Find the cosine of the angle between the vectors $\langle 3, 0, 4\rangle$ and $\langle 0, 12, 5\rangle$.

SKETCH OF SOLUTION. $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{4}{13}$. Note: $\theta \approx 1.258 \approx 72.1°$.

18 (§12.3) Find a unit vector orthogonal to both $\vec{\imath} + \vec{\jmath} - 2\vec{k}$ and $\vec{\imath} - 2\vec{\jmath} + 4\vec{k}$.

SKETCH OF SOLUTION. Do without the cross product: $\langle x, y, z\rangle \cdot \langle 1, 1, -2\rangle = 0$ and $\langle x, y, z\rangle \cdot \langle 1, -2, 4\rangle = 0$ yield two equations in three unknowns. Answer: $\pm\frac{1}{\sqrt{5}}\langle 0, 2, 1\rangle$.

19 (§12.3) (a) Are the vectors $\vec{a} = \langle 2, 5\rangle$ and $\vec{b} = \langle -4, -10\rangle$ orthogonal, parallel or neither? (b) Are the vectors $\vec{v} = \langle 1, -2, 1\rangle$ and $\vec{w} = \langle 0, 1, 2\rangle$ orthogonal, parallel or neither?

SKETCH OF SOLUTION. (a) Parallel: $-2\vec{a} = -2\langle 2, 5\rangle = \vec{b}$. (b) Orthogonal: $\vec{v}\cdot\vec{w} = 0$.

20 (§12.3) For what value(s) of c are the vectors $\langle -6, c, 2\rangle$ and $\langle c, c^2, c\rangle$ orthogonal?

SKETCH OF SOLUTION. $\langle -6, c, 2\rangle \cdot \langle c, c^2, c\rangle = 0$ gives $c^3 - 4c = 0$. Hence, $c = 0, -2, 2$.

21 (§12.3) A cart is pulled up a 20° slope a distance of 10 meters by a horizontal force of 30 newtons. Determine the work.

SKETCH OF SOLUTION. Horizontal force is $\vec{F} = 30\vec{i} + 0\vec{j}$. Displacement vector is $\vec{D} = 10\cos 20^\circ\ \vec{i} + 10\sin 20^\circ\ \vec{j}$. The work is $W = \vec{F}\cdot\vec{D} = 300\ \cos 20^\circ \approx 281.9$ Joules.

22 (§12.3) Prove that the diagonals of a rhombus meet at right angles. Recall, a rhombus is a parallelogram with sides of equal lengths. In particular, a rhombus is determined by vectors $\vec{v}$ and $\vec{w}$ of equal length.

SKETCH OF SOLUTION. Label adjacent sides $\vec{v}$ and $\vec{w}$, where $|\vec{v}| = |\vec{w}|$. Then show that $(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = 0$.

23 (§12.3) Find the angle between a diagonal of a cube and one of its edges.

SKETCH OF SOLUTION. Place the unit cube in the first octant, with one vertex at the origin. Edge: $\langle 1, 0, 0\rangle$; Diagonal: $\langle 1, 1, 1\rangle$. $\cos\theta = 1/\sqrt{3}$ and $\theta \approx 0.9553 \approx 54.7^\circ$.

24 (§12.4) Find a unit vector orthogonal to both $\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{i} - 2\vec{j} + 4\vec{k}$.

SKETCH OF SOLUTION. $[\vec{i} + \vec{j} - 2\vec{k}] \times [\vec{i} - 2\vec{j} + 4\vec{k}] = 0\vec{i} - 6\vec{j} - 3\vec{k}$. Unit vector: $\pm\frac{1}{\sqrt{5}}\langle 0, 2, 1\rangle$.

25 [C] (§12.4) Find the angle between adjacent faces of the regular tetrahedron with vertices $P(0, 0, 1), Q(1, 0, 0), R(0, 1, 0), S(1, 1, 1)$.

SKETCH OF SOLUTION. $\overrightarrow{PQ} = \langle 1, 0, -1\rangle$, $\overrightarrow{PS} = \langle 1, 1, 0\rangle$, $\vec{n_1} = \overrightarrow{PQ} \times \overrightarrow{PS} = \langle 1, -1, 1\rangle$. Also, $\overrightarrow{PR} = \langle 0, 1, -1\rangle$ and $\vec{n_2} = \overrightarrow{PR} \times \overrightarrow{PS} = \langle 1, -1, -1\rangle$.
$\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \dfrac{1}{3}$. Hence, $\theta \approx 1.23 \approx 70.5^\circ$.

26 (§12.4) Find the volume of the parallelepiped determined by the vectors $\vec{a} = \langle 1, 1, 0\rangle$, $\vec{b} = \langle 0, 1, 2\rangle$, and $\vec{c} = \langle -3, 0, 1\rangle$.

SKETCH OF SOLUTION. Use triple scalar product: $\vec{a} \cdot (\vec{b} \times \vec{c}) = -5$. $V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = 5$.

27 (§12.5) Find parametric equations and symmetric equations for the line through the points $A(6, 1, -3)$ and $B(2, 4, 5)$.

SKETCH OF SOLUTION. $\vec{v} = \overrightarrow{AB} = \langle -4, 3, 8\rangle$. Parametric: $x = 6 - 4t, y = 1 + 3t, z = -3 + 8t$. Symmetric: $\dfrac{x-6}{-4} = \dfrac{y-1}{3} = \dfrac{z+3}{8}$.

28 (§12.5) Find the distance from the point $(1, 2, 3)$ to the line $x = t,\ y = 2t - 1,\ z = 3$.

SKETCH OF SOLUTION. $Q(0, -1, 3), R(1, 1, 3)$ are on the line, $\overrightarrow{QR} = \langle 1, 2, 0\rangle$. $P(1, 2, 3), \overrightarrow{QP} = \langle 1, 3, 0\rangle$. If θ is the angle between $\overrightarrow{QR}$ and $\overrightarrow{QP}$, then

$$D = |\overrightarrow{QP}| \sin\theta = \frac{|\overrightarrow{QP}||\overrightarrow{QR}|}{|\overrightarrow{QR}|} \sin\theta = \frac{|\overrightarrow{QP} \times \overrightarrow{QR}|}{|\overrightarrow{QR}|} = \frac{|-\vec{k}|}{|\overrightarrow{QR}|} = 1/\sqrt{5}$$

29 (§12.5) Find an equation of the plane containing the points $P_1(1, 2, 0),\ \ P_2(0, 1, 2)$, and $P_3(2, 0, 4)$.

SKETCH OF SOLUTION. $\overrightarrow{P_1P_2} = \langle -1, -1, 2\rangle,\ \overrightarrow{P_1P_3} = \langle 1, -2, 4\rangle$. Their cross product $\langle 0, 6, 3\rangle$ is normal to the plane. $0(x-1)+6(y-2)+3(z-0) = 0$ gives the equation $2y + z - 4 = 0$.

30 (§12.5) Find parametric equations for the line of intersection of the planes

$$\begin{aligned} \mathcal{P}_1 &: \ x - 2y + 4z = 3 \\ \mathcal{P}_2 &: \ x + y - 2z = 0. \end{aligned}$$

SKETCH OF SOLUTION. The cross product $\langle 0, 6, 3\rangle$ of the normal vectors to the planes is parallel to the line of intersection. Using the common point $(1, -1, 0)$, you obtain parametric equations $x = 1,\ y = -1 + 6t,\ z = 3t$.

31 (§12.5) Find an equation of the plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$.

SKETCH OF SOLUTION. $(1, 3, 0)$ lies on line of intersection. $\vec{v} = \langle 1, 0, -1\rangle \times \langle 0, 1, 2\rangle = \langle 1, -2, 1\rangle$ is a direction vector for the line of intersection. $\vec{w} = \langle 1, 1, -2\rangle$ is also parallel to desired plane. Normal to plane is $\vec{v} \times \vec{w} = \langle 3, 3, 3\rangle$. Hence, $3(x-1) + 3(y-3) + 3z = 0$ or $x + y + z = 4$.

32 (§12.5) Find the distance from the point $(1, 2, 3)$ to the plane $x - 2y + 2z = 4$.

SKETCH OF SOLUTION. $\vec{n} = \langle 1, -2, 2\rangle, P_1 = (1, 2, 3)$, and $P_0 = (4, 0, 0)$ lies in the plane. $\overrightarrow{P_0P_1} = \langle -3, 2, 3\rangle$. $D = |\text{proj}_{\vec{n}} \overrightarrow{P_0P_1}| = \left| \frac{\vec{n} \cdot \overrightarrow{P_0P_1}}{\vec{n} \cdot \vec{n}} \vec{n} \right| = \frac{|\vec{n} \cdot \overrightarrow{P_0P_1}|}{|\vec{n}|} = \frac{1}{3}$.

33 (§12.6) Describe and sketch the surfaces.

$$\begin{aligned} x^2 + y^2 &= 9 \\ z &= x^2 \\ z &= \cos y \end{aligned}$$

SKETCH OF SOLUTION. circular cylinder, parabolic cylinder, cylindrical surface

34 (§12.6) Sketch and identify the surfaces.

$$\begin{aligned} x^2 - y^2 + z^2 &= 1 \\ x^2 - y^2 + z^2 &= 0 \\ x^2 - y^2 - z^2 &= 1 \\ x^2 - y - z^2 &= 0. \end{aligned}$$

SKETCH OF SOLUTION. hyperboloid of one sheet, right circular cone, hyperboloid of two sheets, hyperbolic paraboloid

35 **[C]** (§12.6) Find the volume of the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

SKETCH OF SOLUTION. Ellipse with equation $\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1$ has area πAB. The slice at height z between 0 and c has area $\pi \left(a\sqrt{1 - \dfrac{z^2}{c^2}} \right) \left(b\sqrt{1 - \dfrac{z^2}{c^2}} \right) = \pi ab \left(1 - \dfrac{z^2}{c^2} \right)$. Total volume is $V = \displaystyle\int_{-c}^{c} \pi ab \left(1 - \frac{z^2}{c^2} \right) dz = \frac{4}{3} \pi abc.$

36 (§12.7) Find an equation in cylindrical coordinates for the rectangular equation

$$x^2 + y^2 = 4y.$$

Sketch the surface.

SKETCH OF SOLUTION. $r^2 = 4r \sin\theta$, or $r = 4 \sin\theta$.

37 (§12.7) Find an equation in spherical coordinates for the rectangular equation

$$x^2 + y^2 + z^2 - 4z = 0.$$

Sketch the surface.

SKETCH OF SOLUTION. $\rho^2 - 4\rho\cos\phi = 0$, or $\rho = 4\cos\phi$.

38 (§12.7) Convert the point $(r, \theta, z) = (3, \frac{-\pi}{6}, 4)$ from cylindrical to spherical coordinates.

SKETCH OF SOLUTION. $(\rho, \theta, \phi) = (5, -\pi/6, \arcsin(3/5)) \approx (5, -\pi/6, 0.64)$.

39 (§12.7) Convert the point $(\rho, \theta, \phi) = (9, \frac{\pi}{4}, \frac{\pi}{4})$ from spherical to cylindrical coordinates.

SKETCH OF SOLUTION. $(r, \theta, z) = (9/\sqrt{2}, \pi/4, 9/\sqrt{2})$

40 (§12.7) Sketch the solid given by the spherical coordinates

$$0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/6,\ 0 \le \rho \le 2\sec\phi$$

SKETCH OF SOLUTION. $\rho = 2/\cos\phi$, or $z = \rho\cos\phi = 2$, a solid cone.

41 (§13.1) Show that the point $(-2, 4, -8)$ is on the twisted cubic $x = t$, $y = t^2$, $z = t^3$, but that $(-2, 4, 8)$ is not.

SKETCH OF SOLUTION. $t = -2$ for first point; no t works for second point.

42 (§13.1) Sketch the curve $\vec{r} = \langle t^4, t^5 \rangle$.

SKETCH OF SOLUTION. art

43 (§13.1) Compute the limit,

$$\lim_{t\to 0} \left\langle \frac{e^t - 1}{t}, \frac{\sin(t)}{t}, 1 + t^3 \right\rangle$$

if it exists.

SKETCH OF SOLUTION. L'Hospital's Rule gives $\langle 1, 1, 1 \rangle$.

44 (§13.1) Find a vector valued function $\vec{r}$ which traces out the curve of intersection of the surfaces,

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1, \quad y > 0$$

and

$$z = x^2.$$

Sketch the curve.

SKETCH OF SOLUTION. Let $x = t$: $\vec{r}(t) = \left\langle t, 2\sqrt{1 - t^2 - \frac{t^4}{9}}, t^2 \right\rangle$

45 (§13.2) Sketch the plane curve $\vec{r}(t) = 3\sin t\,\vec{i} + 4\cos t\,\vec{j}$ and find $\vec{r}\,'(t)$. Sketch the position vector $\vec{r}(t)$ and the tangent vector $\vec{r}\,'(t)$ for $t = \pi/3$.

SKETCH OF SOLUTION. $\vec{r}\,'(t) = 3\cos t\,\vec{i} - 4\sin t\,\vec{j}$, $\vec{r}\left(\frac{\pi}{3}\right) = \left\langle \frac{3\sqrt{3}}{2}, 2 \right\rangle$, $\vec{r}\,'\left(\frac{\pi}{3}\right) = \left\langle \frac{3}{2}, -2\sqrt{3} \right\rangle$.

46 (§13.2) Find parametric equations for the tangent line to the curve

$$x = 2\sqrt{t},\ y = t^2,\ z = \ln t$$

at the point $(2, 1, 0)$.

SKETCH OF SOLUTION. $x' = 1/\sqrt{t}, y' = 2t, z' = 1/t$. The point $(2, 1, 0)$ corresponds to $t = 1$. Thus, $x'(1) = 1, y'(1) = 2, z'(1) = 1$ and

$$x = 2 + t,\ y = 1 + 2t,\ z = t.$$

47 (§13.2) Solve the initial value problem,

$$\frac{d\vec{r}}{dt} = \langle 2t, e^t, 4t^3 \rangle, \quad \vec{r}(0) = \langle 1, 3, 0 \rangle.$$

SKETCH OF SOLUTION. $\vec{r}(t) = \langle t^2 + 1, e^t + 2, t^4 \rangle$

48 (§13.3) Let a and b be positive constants. Compute $\vec{T}, \vec{N}, \vec{B}$, and κ for the helix,

$$\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle.$$

SKETCH OF SOLUTION. $\vec{r}\,' = \langle -a\sin t, a\cos t, b \rangle$, $|\vec{r}\,'| = \sqrt{a^2 + b^2}$,
$\vec{T} = \left\langle -\frac{a\sin t}{\sqrt{a^2+b^2}}, \frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right\rangle$. $\vec{T}\,' = \left\langle -\frac{a\cos t}{\sqrt{a^2+b^2}}, -\frac{a\sin t}{\sqrt{a^2+b^2}}, 0 \right\rangle$,
$|\vec{T}\,'| = \frac{a}{\sqrt{a^2+b^2}}$, $\vec{N} = \langle -\cos t, -\sin t, 0 \rangle$.
$\vec{B} = \vec{T} \times \vec{N} = \left\langle \frac{b\sin t}{\sqrt{a^2+b^2}}, -\frac{b\cos t}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}} \right\rangle$.
$\kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{a}{a^2+b^2}$.

49 (§13.3) The cornu spiral (http://mathworld.wolfram.com/CornuSpiral.html) $r(t)$ satisfies $r(0) = 0$ and

$$\frac{d\vec{r}}{dt} = \left\langle \cos\left(\frac{t^2}{2}\right), \sin\left(\frac{t^2}{2}\right) \right\rangle.$$

Since $|\frac{d\vec{r}}{dt}| = 1$, this curve is parameterized by arc length, $t = s$. Show, for $t > 0$, that its curvature equals its arc length. (Up to translation, rotation, and reflection, this is the only curve with this property).

Sketch of solution. $\vec{T}(s) = \left\langle \cos\left(\frac{s^2}{2}\right), \sin\left(\frac{s^2}{2}\right) \right\rangle$, $\frac{d\vec{T}}{ds} = \left\langle -s\sin\left(\frac{s^2}{2}\right), s\cos\left(\frac{s^2}{2}\right) \right\rangle$.
For $s > 0$, $\kappa(s) = \left|\frac{d\vec{T}}{ds}\right| = s$.

50 (§13.3) Reparameterize the curve

$$\vec{r}(t) = \langle e^t \cos(t), e^t \sin(t), e^t \rangle$$

by arc length.

Sketch of solution. $|\vec{r}\,'(t)|^2 = 3e^{2t}$. Hence, $\frac{ds}{dt} = \sqrt{3}e^t$. Assuming s is measured from $t = 0$, $s = \sqrt{3}e^t - \sqrt{3}$, $t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$.

$$\vec{r}(s) = \left(\frac{s}{\sqrt{3}} + 1\right)\left\langle \cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), 1 \right\rangle.$$

51 (§13.3) Find the curvature of $f(x) = \cos x$ at the point $(0, 1)$.

Sketch of solution. $f'(x) = -\sin x, f''(x) = -\cos x$.

$$\kappa(0) = \frac{|f''(0)|}{[1 + (f'(0)^2]^{3/2}} = \frac{|-1|}{[1+0]} = 1$$

52 (§13.3) Find the equation of the osculating circle for the parabola $y = x - \frac{x^2}{4}$ at the point $(2, 1)$.

Sketch of solution. $y' = 1 - x/2, y'' = -1/2, \kappa(2) = \frac{1}{2}$. Radius 2 and center $(2, -1)$. $(x-2)^2 + (y+1)^2 = 4$.

53 (§13.4) For the curve

$$\vec{r}(t) = \left\langle t, t^2 + 6t, \frac{1}{3}t^3 - 9t \right\rangle$$

verify that velocity and acceleration are perpendicular when $t = 1$. Determine all positions at which the velocity is perpendicular to acceleration. Is velocity ever parallel to acceleration?

SKETCH OF SOLUTION. $\vec{v} \cdot \vec{a} = 2(t-1)(t-2)(t+3) = 0$ implies $t = 1, 2, -3$; No.

54 (§13.4) Compute the tangential and normal components of acceleration for the helix

$$\vec{r}(t) = \langle a \cos t, a \sin t, bt \rangle.$$

SKETCH OF SOLUTION. $\vec{r}\,' = \langle -a \sin t, a \cos t, b \rangle$, $v = |\vec{r}\,'| = \sqrt{a^2 + b^2}$, $a_{\vec{T}} = v' = 0$. From a previous problem, $\kappa = \dfrac{a}{a^2 + b^2}$ and hence $a_{\vec{N}} = \kappa v^2 = \dfrac{a}{a^2 + b^2}(a^2 + b^2) = a$.

55 (§13.4) Find the tangential and normal components of the acceleration vector at $t = 1$ for

$$\vec{r} = t\vec{i} + \frac{1}{t}\vec{j}.$$

SKETCH OF SOLUTION. $\vec{v} = \vec{r}\,' = \vec{i} - \dfrac{1}{t^2}\vec{j}$, $\vec{r}\,'(1) = \vec{i} - \vec{j}$, $\vec{a} = \vec{r}\,'' = \dfrac{2}{t^3}\vec{j}$, $\vec{r}\,''(1) = 2\vec{j}$, $\vec{T} = \dfrac{t^2\vec{i} - \vec{j}}{\sqrt{t^4 + 1}}$, $\vec{T}(1) = \dfrac{\vec{i} - \vec{j}}{\sqrt{2}}$, $\vec{N} = \dfrac{\vec{i} + t^2\vec{j}}{\sqrt{t^4 + 1}}$, $\vec{N}(1) = \dfrac{\vec{i} + \vec{j}}{\sqrt{2}}$, $a_{\vec{T}} = \vec{a} \cdot \vec{T} = -\sqrt{2}$, $a_{\vec{N}} = \vec{a} \cdot \vec{N} = \sqrt{2}$.

56 (§14.1) Let $u(x, y) = x^2 - y^2$. Draw a contour map of u.

SKETCH OF SOLUTION. Level curves are hyperbolas (and the two lines $y = \pm x$)

57 (§14.1) Sketch and describe the domain of the function

$$f(x, y, z) = \log(1 - 4x^2 - 9y^2 - z^2).$$

SKETCH OF SOLUTION. Interior of ellipsoid $\dfrac{x^2}{(1/2)^2} + \dfrac{y^2}{(1/3)^2} + z^2 = 1$.

58 (§14.1) Describe the level surfaces for the function

$$f(x, y, z) = x^2 + y^2 - z^2.$$

Sketch of solution. Set $x^2 + y^2 - z^2 = k$. For $k = 0$, cone. For $k > 0$, hyperboloid of one sheet. For $k < 0$, hyperboloid of two sheets.

59 (§14.2) Let

$$f(x, y) = \frac{x^3 y}{x^6 + y^2}, \quad (x, y) \neq (0, 0).$$

Evaluate the limit

$$\lim_{(x,y)\to(0,0)} f(x, y).$$

Sketch of solution. The limit does not exist. Along $x = 0$, limit is 0. Along $y = x^3$, limit is $1/2$.

60 (§14.2) Let

$$f(x, y) = \frac{e^{x^2+y^2} - 1}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

By converting to polar coordinates, show that

$$\lim_{(x,y)\to(0,0)} f(x, y) = 1.$$

Is it possible to define $f(0, 0)$ so as to make f continuous at $(0, 0)$?

Sketch of solution. $\lim_{r^2\to 0} \frac{e^{r^2} - 1}{r^2} = \lim_{t\to 0^+} \frac{e^t - 1}{t} = 1$. Define $f(0, 0) = 1$.

61 (§14.2) Let

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad x^2 + y^2 > 0.$$

Compute the iterated limits:

$$\lim_{x\to 0} \lim_{y\to 0} f(x, y) \text{ and } \lim_{y\to 0} \lim_{x\to 0} f(x, y).$$

Sketch of solution. $\lim_{x\to 0} \lim_{y\to 0} f(x, y) = \lim_{x\to 0} \left(\frac{x^2}{x^2}\right) = 1$,
$\lim_{y\to 0} \lim_{x\to 0} f(x, y) = \lim_{y\to 0} \left(\frac{-y^2}{y^2}\right) = -1$.

62 **[C]** (§14.2) Determine the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \sin\left(\frac{1}{x^2 + y^2}\right),$$

if it exists.

Sketch of solution. Limit does not exist. Along path $y = 0$, limit is 0. Along path $x = y$, the function $\frac{1}{2}\sin(\frac{1}{2x^2})$ oscillates between $-1/2$ and $1/2$.

63 (§14.3) Compute the second partial derivatives of the function

$$f(x, y) = \tan^{-1}\left(\frac{x}{y}\right).$$

Sketch of solution. $f_{xx} = \dfrac{-2xy}{(x^2+y^2)^2}$, $f_{yy} = \dfrac{2xy}{(x^2+y^2)^2}$, $f_{xy} = f_{yx} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$.

64 (§14.3) Assuming that the equation

$$-x^2y + z^3x + z = 1$$

(implicitly) defines z as a function of (x, y) near the point $(1, 1, 1)$, find the value of $\frac{\partial z}{\partial x}$ at the point $(1, 1, 1)$.

Sketch of solution. $z_x = \dfrac{2xy - z^3}{3z^2x + 1}$. At $(1, 1, 1)$, $z_x = \dfrac{1}{4}$.

65 (§14.3) Verify that the function

$$u = x^2 - y^2$$

satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Sketch of solution. Proof: $u_{xx} = 2$ and $u_{yy} = -2$.

66 (§14.3) Could

$$f_x = x + 3y$$
$$f_y = 4x + y$$

be the partial derivatives of a function $f(x, y)$ defined in the plane?

Sketch of solution. No. From f_x you obtain $f_{xy} = 3$ and from f_y you obtain $f_{yx} = 4$.

67 (§14.3) Could

$$f_x = \frac{x}{x^2 + y^2 + 1}$$
$$f_y = \frac{y}{x^2 + y^2 + 1}$$

be the partial derivatives of a function f?

SKETCH OF SOLUTION. Yes. $f(x, y) = \frac{1}{2}\ln(x^2 + y^2 + 1) + C$.

68 (§14.3) Let

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{when } (x, y) \neq (0, 0) \\ 0 & \text{when } (x, y) = (0, 0). \end{cases}$$

Find $f_x(0, 0)$ and $f_y(0, 0)$. Is f continuous at $(0, 0)$?

SKETCH OF SOLUTION. Using the definition of partial derivative, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. But, f is not continuous: along $y = 0$ limit is 0, along $y = x$ limit is $1/2$.

69 (§14.4) Sketch the surface $f(x, y) = 25 - x^2 - y^2$. Then find the equation of the tangent plane to the surface at the point $(3, 1, 15)$.

SKETCH OF SOLUTION. $6x + 2y + z = 35$.

70 (§14.4) Find the equation of the tangent plane to the surface,

$$z = \frac{1}{2}\ln(x^2 + y^2 + 3)$$

at the point $(2, 3, \ln(4))$.

SKETCH OF SOLUTION. $2x + 3y - 16z = 13 - 16\ln 4$.

71 (§14.4) Let $f(x, y) = \sqrt{x^2 + y^2}$.

(a) Find the linearization of f at the point $(4, 3)$; and

(b) use it to approximate $f(4.1, 2.8)$.

SKETCH OF SOLUTION. (a) $L(x, y) = 5 + \frac{4}{5}(x - 4) + \frac{3}{5}(y - 3)$. (b) 4.96. Note: exact value is 4.9648766....

72 (§14.4) Compute the differential of $z = xe^{xy^2}$.

SKETCH OF SOLUTION. $dz = e^{xy^2}(1 + xy^2)\, dx + e^{xy^2}(2x^2y)\, dy$.

73 (§14.4) If $z = x^2 - xy + 3y^2$ and (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz and dz.

SKETCH OF SOLUTION. $dx = -0.04$ and $dy = 0.05$. $dz = (7)(-0.04) + (-9)(0.05) = -0.73$, and $\Delta z = -0.7189$.

74 (§14.5) Suppose

$$w = x^2 + y^2, \quad x = 2st, \quad y = s^2 - t^2.$$

Use the chain rule to compute, $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial s}$.

SKETCH OF SOLUTION. $\frac{\partial w}{\partial t} = 4s^2t + 4t^3$; $\frac{\partial w}{\partial s} = 4st^2 + 4s^3$

75 (§14.5) Suppose f is a differentiable function of x and y, and

$$g(u, v) = f(e^u \cos(v), e^u \sin(v)).$$

If $f_x(1, 0) = 2$ and $f_y(1, 0) = 5$, find $g_u(0, 0)$ and $g_v(0, 0)$.

SKETCH OF SOLUTION. $g_u(0, 0) = 2; g_v(0, 0) = 5$.

76 (§14.5) The equation

$$x - z = \ln(yz)$$

(implicitly) defines z as a function of (x, y) near $(1, 1, 1)$. Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 1, 1)$.

SKETCH OF SOLUTION. $\dfrac{\partial z}{\partial x} = \dfrac{z}{1+z} = \dfrac{1}{2}; \dfrac{\partial z}{\partial y} = \dfrac{-z}{y(1+z)} = -\dfrac{1}{2}$.

77 (§14.5) The equation

$$x - z = \ln(yz)$$

defines z as a function of (x, y) near $(1, 1, 1)$. Suppose now that $x = x(t)$ and $y = y(t)$ are differentiable and $x(0) = y(0) = 1$, $x'(0) = 2$ and $y'(0) = 3$. Find $\frac{dz}{dt}$ at $t = 0$.

SKETCH OF SOLUTION. Use the previous problem,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{z}{1+z}x' + \frac{-z}{y(1+z)}y'.$$

$z = 1$ when $t = 0$, so $\frac{dz}{dt} = 1 - 3/2 = -1/2$.

78 (§14.6) Find the gradients of (a) $f(x, y) = x^3y^{-2}$ and (b) $g(x, y, z) = x^2e^{2z} + ye^{y^2z}$.

SKETCH OF SOLUTION. (a)$(3x^2y^{-2})\vec{i} + (-2x^3y^{-3})\vec{j}$
(b)$(2xe^{2z})\vec{i} + (e^{y^2z} + 2y^2ze^{y^2z})\vec{j} + (2x^2e^{2z} + y^3e^{y^2z})\vec{k}$.

79 (§14.6) Let $f(x,y) = e^{x^2+y^2}$, $P = (-1,1)$, and $\vec{u} = \frac{1}{\sqrt{5}}(\vec{\imath} + 2\vec{\jmath})$. Compute the directional derivative of f in the direction $\vec{u}$ at the point P.

SKETCH OF SOLUTION. $D_{\mathbf{u}}f(x,y) = \langle 2xe^{x^2+y^2}, 2ye^{x^2+y^2}\rangle \cdot \langle 1/\sqrt{5}, 2/\sqrt{5}\rangle$; $D_{\mathbf{u}}f(-1,1) = 2e^2/\sqrt{5}$.

80 (§14.6) Let $f(x,y) = xy$. Find the gradient of f. Sketch some level curves, $f(x,y) = \text{constant}$. Indicate both grad(f) and the tangent line to the curve at some points.

SKETCH OF SOLUTION. $\nabla f(x,y) = \langle y, x\rangle$. At $(4,1)$ on level curve $xy = 4, \nabla f(4,1) = \langle 1,4\rangle$ of slope 4. The tangent line at $(4,1)$ has slope $-1/4$. Gradient orthogonal to level curve.

81 (§14.6) Let $V(x,y,z) = x^2 - xy + xyz$. Find the rate of change of V in the direction $\frac{1}{\sqrt{3}}\langle 1,-1,1\rangle$ at the point $(1,1,1)$. In which direction(s) does V change most rapidly at $(1,1,1)$? What is the maximum rate of change of V at $(1,1,1)$?

SKETCH OF SOLUTION. $D_{\mathbf{u}}V(x,y,z) = \langle 2x-y+yz, -x+xz, xy\rangle\cdot\mathbf{u}$. $D_{\mathbf{u}}V(1,1,1) = \langle 2,0,1\rangle \cdot \langle 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\rangle = \sqrt{3}$. V changes most rapidly in the direction of the gradient, $\nabla V = \langle 2,0,1\rangle$. The maximum rate of change of V at $(1,1,1)$ is $|\nabla V| = |\langle 2,0,1\rangle| = \sqrt{5}$.

82 (§14.6) Show that the level curves for $f(x,y) = x^2 - y^2$ and $g(x,y) = xy$ are orthogonal.

SKETCH OF SOLUTION. For $x^2 - y^2 = C_1, y' = x/y$. For $xy = C_2, y' = -y/x$.

83 (§14.6) Find the direction of steepest ascent at the point $(-1,0)$ for the surface $z = f(x,y) = x^2 - y^2$? Illustrate this with a graph.

SKETCH OF SOLUTION. $\nabla f(-1,0) = \langle -2, 0\rangle$.

84 (§14.6) Find an equation of the tangent plane to the surface $z^2 - 2x^2 - 2y^2 = 12$ at the point $(1,-1,4)$.

SKETCH OF SOLUTION. $x - y - 2z + 6 = 0$

85 (§14.7) Suppose $(0,0)$ is a critical point of a function h with continuous second partial derivatives. In each case, what can you say about the nature of the critical point of h at $(0,0)$?

$$\begin{aligned}
&(1)\ h_{xx}(0,0) = -3, \quad h_{xy}(0,0) = 2, \quad h_{yy}(0,0) = -2\\
&(2)\ h_{xx}(0,0) = 3, \quad h_{xy}(0,0) = 2, \quad h_{yy}(0,0) = 2\\
&(3)\ h_{xx}(0,0) = 1, \quad h_{xy}(0,0) = 2, \quad h_{yy}(0,0) = 2\\
&(4)\ h_{xx}(0,0) = 2, \quad h_{xy}(0,0) = 2, \quad h_{yy}(0,0) = 2
\end{aligned}$$

SKETCH OF SOLUTION. Case 1: $D = 2 > 0$ and $h_{xx} < 0$ implies local maximum. Case 2: $D = 2 > 0$ and $h_{xx} > 0$ implies local minimum. Case 3: $D = -2 < 0$ implies saddle point. Case 4: $D = 0$. Second Derivative Test gives no information.

86 (§14.7) Find and classify the critical points of

$$f(x, y) = \frac{1}{2}x^2 + \frac{1}{3}y^3 - xy$$

SKETCH OF SOLUTION. $(0, 0)$ is saddle point, $(1, 1)$ is local minimum.

87 (§14.7) Find the maximum and minimum (greatest and least values) of the function

$$f(x, y) = \frac{1}{2}x^2 + \frac{1}{3}y^3 - xy$$

in the domain $0 \le x \le 2$, $0 \le y \le 2$.

SKETCH OF SOLUTION. From the previous problem, $f(0, 0) = 0$, and $f(1, 1) = -1/6$. Now analyze the four boundary line segments. Along $y = 2, 0 \le x \le 2 :$ $f(0, 2) = 8/3, f(2, 2) = 2/3$. Along $x = 0, 0 \le y \le 2 : f(0, 2) = 8/3, f(0, 0) = 0$. Along $y = 0, 0 \le x \le 2 : f(0, 0) = 0, f(2, 0) = 2$. Along $x = 2, 0 \le y \le 2 : f(2, 2) = 2/3, f(2, 0) = 2, f(2, \sqrt{2}) = 2 - 4\sqrt{2}/3$. Conclusion: Maximum $f(0, 2) = 8/3$, Minimum $f(1, 1) = -1/6$.

88 (§14.7) Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$ by (i) minimizing $f(x, y) = (x - 1)^2 + (y - 2)^2 + (1 - x + y)^2$; (ii) considering the magnitude of the projection of the point onto the plane; and (iii) using the method of Lagrange.

SKETCH OF SOLUTION. (i) Minimize square of distance $f(x, y) = (x - 1)^2 + (y - 2)^2 + (1 - x + y)^2$. Setting $f_x = f_y = 0$, you obtain $(x, y) = (5/3, 4/3)$. This is a minimum by Second Derivative Test: $D = 12 > 0$ and $f_{xx} = 4 > 0$. Finally, distance is $\frac{2\sqrt{3}}{3}$.

(ii) $\vec{n} = \langle 1, -1, 1 \rangle$ normal to plane. $P_1 = (1, 2, 3)$; $P_0 = (4, 0, 0)$ lies in plane, $\vec{b} = \overrightarrow{P_0P_1} = \langle -3, 2, 3 \rangle$. Distance is $|\text{proj}_{\vec{n}}\vec{b}| = \frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|} = \frac{2\sqrt{3}}{3}$.

(iii) Minimize $f(x, y, z) = (x-1)^2+(y-2)^2+(z-3)^2$ subject to $g(x, y, z) = x - y + z - 4 = 0$. Setting $\nabla f = \lambda \nabla g$ you obtain $x = 5/3, y = 4/3, z = 11/3$ and $\lambda = 4/3$.

89 (§14.7) Explain how we know $V = xyz$ has a maximum subject to the constraint $2x + 2y + z = 1$ and $x, y, z \ge 0$ at a point (a, b, c) where $a, b, c \ge 0$.

SKETCH OF SOLUTION. Use Extreme Value Theorem for $V = xy(1 - 2x - 2y)$ defined on triangle D with vertices $(0, 0), (1/2, 0), (0, 1/2)$.

90 (§14.7 or 14.8) Find the extreme values of

$$f(x,y) = 3x^2 + 2y^2 - 4y + 2$$

subject to the constraint $x^2 + y^2 \leq 16$.

SKETCH OF SOLUTION. On interior, critical point $(0,1)$. On boundary, analyze $f(x,y) = g(y) = 3(16 - y^2) + 2y^2 - 4y + 2, -4 \leq y \leq 4$ to obtain $(\pm 2\sqrt{3}, -2), (0, \pm 4)$. Maximum is 54 at $(\pm 2\sqrt{3}, -2)$, minimum is 0 at $(0,1)$. Or use Lagrange multipliers for boundary.

91 (§14.8) Use Lagrange multipliers to find the maximum of $V = xyz$ subject to the constraint $2x + 2y + z = 1$ and $x, y, z \geq 0$.

SKETCH OF SOLUTION. $V = 1/108$ for $x = y = 1/6, z = 1/3$.

92 (§15.1) Express the volume of the solid which lies below the surface $z = x^2 + 2y$ and above the rectangle $[0,2] \times [0,4]$ as a double integral. Use a Riemann sum with $n = m = 2$ and (i) choosing the sample points to be the upper left corners and (ii) choosing the midpoints as the sample points to approximate this volume.

SKETCH OF SOLUTION. (i) $[f(0,2) + f(0,4) + f(1,2) + f(1,4)] \cdot 2 = 52$. (ii) $[f(1/2,1) + f(1/2,3) + f(3/2,1) + f(3/2,3)] \cdot 2 = 42$. Note: Exact volume is $128/3 \approx 42.67$.

93 (§15.2) Evaluate the iterated integral

$$\int_0^2 \int_0^4 (x^2 + 2y)\, dy\, dx.$$

SKETCH OF SOLUTION. $I = \displaystyle\int_0^2 [4x^2 + 16]\, dx = \frac{128}{3}.$

94 (§15.3) Express the area of the region bounded by $y = x^5$ and $y = x^2$ as an iterated integral.

SKETCH OF SOLUTION. $\displaystyle\int_0^1 \int_{\sqrt{y}}^{\sqrt[5]{y}} 1\, dx\, dy = \int_0^1 \int_{x^5}^{x^2} 1\, dy\, dx = \frac{1}{6}.$

95 (§15.3) Calculate the double integral,

$$\iint_R xy\, dA,$$

where R is the region bounded by $y = 0$, $x = 2$ and $y = x^3$.

SKETCH OF SOLUTION. $\int_0^8 \int_{\sqrt[3]{y}}^2 xy\,dx\,dy = \int_0^2 \int_0^{x^3} xy\,dy\,dx = 16.$

96 (§15.3) Calculate the double integral,

$$\iint_D y dA$$

where D is the region bounded by $y = x^2 - 2$ and $y = x$.

SKETCH OF SOLUTION. $\int_{-1}^2 \int_{x^2-2}^x y\,dy\,dx = -\frac{9}{5}.$

97 (§15.3) Sketch the region of integration and change the order of integration.

(a) $\int_0^4 \int_0^y f(x,y)\,dx\,dy$

(b) $\int_{-1}^1 \int_{x^2}^1 f(x,y)\,dy\,dx$

(c) $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$

SKETCH OF SOLUTION. (a) $\int_0^4 \int_x^4 f(x,y)\,dy\,dx$

(b) $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$ (c) $\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$

98 (§15.3) Compute the double integral,

$$\iint_R e^{x^2} dA$$

where R is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

SKETCH OF SOLUTION. $\int_0^1 \int_0^x e^{x^2}\,dy\,dx = \frac{e-1}{2}.$

99 (§15.3) Compute the iterated integral,

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy$$

by first changing the order of integration.

SKETCH OF SOLUTION. $\int_0^1 \int_0^{x^2} e^{x^3}\,dy\,dx = \frac{e-1}{3}$.

100 (§15.3) Evaluate the iterated integral

$$\int_0^1 \int_x^1 (1-y^2)^{\frac{1}{3}} dy dx$$

by first changing the order of integration.

SKETCH OF SOLUTION. $\int_0^1 \int_0^y (1-y^2)^{1/3}\,dx\,dy = \frac{3}{8}$.

101 (§15.3) Express the volume of the solid bounded by the cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as an iterated integral. Evaluate the integral.

SKETCH OF SOLUTION. Look at the volume in the first octant. $16 \int_0^1 \int_0^x \sqrt{1-x^2}\,dy\,dx = \frac{16}{3}$

102 (§15.4) Express

$$\iint_R xy^3\,dA$$

as an iterated integral in polar coordinates, where R is the set $x^2 + y^2 \leq 1$, $x, y \geq 0$.

SKETCH OF SOLUTION. $\int_0^{\pi/2} \int_0^1 r^5 \cos\theta \sin^3\theta\,dr\,d\theta = \frac{1}{24}$.

103 (§15.4) Suppose $0 < R < \rho$. (a) Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = \rho^2$ and the cylinder $x^2 + y^2 = R^2$. (b) Find the volume inside the sphere, but outside the cylinder.

SKETCH OF SOLUTION. (a) $\int_0^{2\pi} \int_0^R 2\sqrt{\rho^2 - r^2}\,r\,dr\,d\theta = \frac{4\pi}{3}[\rho^3 - (\rho^2 - R^2)^{3/2}]$. If $R = \rho$, volume of sphere. (b) $\int_0^{2\pi} \int_R^\rho 2\sqrt{\rho^2 - r^2}\,r\,dr\,d\theta = \frac{4\pi}{3}(\rho^2 - R^2)^{3/2}$. If $R = 0$, volume of sphere. Sum of answers is volume of sphere.

104 (§15.4) Evaluate

$$\iint_R \frac{1}{\sqrt{x^2 + y^2}}\,dx\,dy,$$

where R is the disc of radius one with center $(1, 0)$, by switching to polar coordinates.

Sketch of solution. Region has equation $r = 2\cos\theta, -\pi/2 \le \theta \le \pi/2$. The integral is $\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{1}{r}\, r\, dr\, d\theta = 4$.

105 (§15.4) Compute

$$\iint_{D_a} e^{-(x^2+y^2)} dA,$$

where D_a is the disc $x^2 + y^2 \le a^2$. Let a tend to infinity and conclude that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dA = \pi.$$

Then find

$$\int_{-\infty}^{\infty} e^{-x^2} dx.$$

Sketch of solution. Define $I_a = \iint_{D_a} e^{-(x^2+y^2)} dA = \int_0^{2\pi}\int_0^a e^{-r^2}\, r\, dr\, d\theta$ $= \pi(1 - e^{-a^2})$. Then $\lim_{a\to\infty} I_a = \pi$. For the challenge, write

$$\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dx\, dy = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2$$

which implies that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

106 (§15.5) A thin metal plate is bounded by the curves $y = x$ and $y = x^2$. Its density at (x, y) is given by xy^2. Find its center of mass.

Sketch of solution. mass $= \int_0^1\int_{x^2}^x xy^2\, dy\, dx = \frac{1}{40}$, $M_x = \int_0^1\int_{x^2}^x xy^3\, dy\, dx = \frac{1}{60}$, $M_y = \int_0^1\int_{x^2}^x x^2y^2\, dy\, dx = \frac{1}{54}$. Answer: $(\bar{x}, \bar{y}) = (20/27, 2/3)$.

107 (§15.6) Find the area of the surface

$$z = \frac{2}{3}\sqrt{2}\, x^{3/2} + y, \quad 0 \le x \le 1, \quad -1 \le y \le 1.$$

Sketch of solution. Area $= \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA = \int_{-1}^1\int_0^1 \sqrt{2x + 1 + 1}\, dx\, dy$ $= \frac{4(4 - \sqrt{2})}{3}$.

108 (§15.6) Find the surface area of the part the plane $z = ax + by + c$ which lies in the cylinder $x^2 + y^2 = R^2$.

SKETCH OF SOLUTION. $\displaystyle\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\sqrt{a^2+b^2+1}\,dy\,dx = \int_0^{2\pi}\int_0^R \sqrt{a^2+b^2+1}\,r\,dr\,d\theta$ $= \pi R^2\sqrt{a^2+b^2+1}$.

109 (§15.6) Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ which lies above the xy-plane.

SKETCH OF SOLUTION. $\displaystyle\int\int_D \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$ $= \frac{\pi}{6}(37\sqrt{37}-1)$.

110 (§15.7) Evaluate

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} dz\,dy\,dx$$

and interpret as the volume of a solid.

SKETCH OF SOLUTION. Tetrahedron with vertices at $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$. Volume is $1/6$.

111 (§15.7) Let S denote the region bounded by $y^2+z^2 = 1$, $x+z = 2$, and $x = 0$. Evaluate the integrals,

$$\text{(a)} \quad \iiint_S \sqrt{1-y^2}\,dV,$$

and

$$\text{(b)} \quad \iiint_S y\,dV.$$

SKETCH OF SOLUTION. (a) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{2-z}\sqrt{1-y^2}\,dx\,dz\,dy = \frac{16}{3}$ (b) 0, by symmetry.

112 (§15.7) Express the volume of the region common to the cylinders $x^2 + y^2 \le a^2$ and $x^2+z^2 \le a^2$ (here $a > 0$) as a triple integral. Evaluate.

SKETCH OF SOLUTION. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 1\,dz\,dy\,dx = \frac{16a^3}{3}$.

113 (§15.8) Evaluate $\displaystyle\iiint_E xy\,dV$, where E is the solid that lies within the cylinder $x^2+y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = x^2+y^2$.

Sketch of solution. $\int_0^{2\pi}\int_0^1\int_0^r (r\cos\theta)(r\sin\theta)r\,dz\,dr\,d\theta = 0$. (also by symmetry)

114 (§15.8) Evaluate

$$\iiint_B \frac{e^{-(x^2+y^2+z^2)}}{\sqrt{x^2+y^2+z^2}}dV,$$

where B is the solid $1 \le x^2+y^2+z^2 \le 4$, $x, y, z \ge 0$.

Sketch of solution. $\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \frac{e^{-\rho^2}}{\rho}(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi = \frac{\pi(e^{-1}-e^{-4})}{4}$.

115 (§15.8) Find the center of mass of a solid hemisphere of radius 2 if the density at any point is proportional to its distance from the base.

Sketch of solution. Place the center at the base $(0,0,0)$.
$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (K\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 4\pi K$.
$M_{xz} = M_{yz} = 0$, $M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (K\rho^4\cos^2\phi)\sin\phi\,d\rho\,d\phi\,d\theta = \frac{64K\pi}{15}$.
Therefore, $(\overline{x},\overline{y},\overline{z}) = (0,0,16/15)$.

116 (§15.9) Find

$$\iint_E \sin(x+y)\cos(x-y)dA,$$

where E is the region bounded by $y = x-2$, $y = x+4$, $y = -x$ and $y = -x+2$.

Sketch of solution. $\int_{-4}^2\int_0^2 \sin u\cos v(1/2)\,du\,dv = (1-\cos 2)(\sin 2+\sin 4)/2 \approx 0.1079776$.

117 (§15.9) Find the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

(here $a, b, c > 0$) by making an appropriate change of variable.

Sketch of solution. Double integral: $V = \int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} 2c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dx$
$= \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 2c\sqrt{1-u^2-v^2}(ab)\,dv\,du = \frac{4\pi}{3}abc$, using $u = x/a$, $v = y/b$.
Triple integral: $V = \int\int\int_R 1\,dz\,dy\,dx = \int\int\int_S a\,b\,c\,dw\,dv\,du$. R is ellipsoidal region, S is spherical region under transformation $u = x/a$, $v = y/b$, $w = z/c$.

118 (§15.9) Let R denote the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$. Use the transformation $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$ to evaluate the integral $\iint_R (4x+8y)\,dA$.

SKETCH OF SOLUTION. $\displaystyle\int_{-4}^{4}\int_{0}^{8}(3v-5u)\,\frac{1}{4}\,dv\,du = 192.$

119 (§16.1) Let $f(x,y) = xy$. Sketch the gradient vector field of f together with a contour map of f. Explain how they are related.

SKETCH OF SOLUTION. gradient vectors are perpendicular to level curves.

120 (§16.1) Sketch the vector field

$$\vec{F}(x,y) = \frac{-y\vec{\imath} + x\vec{\jmath}}{x^2+y^2}, \quad x^2+y^2 > 0.$$

SKETCH OF SOLUTION. At each point (x,y), $\vec{F}(x,y)$ is perpendicular to $\langle x,y\rangle$. The vectors point in a counterclockwise direction about the origin.

121 (§16.2) Evaluate

$$\int_C xy\,ds,$$

where C is the boundary of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

SKETCH OF SOLUTION. Let C_1 be the segment from $(1,0)$ to $(1,1)$, and so on. Then $\displaystyle\int_C xy\,ds = 1/2 + 1/2 + 0 + 0 = 1.$

122 (§16.2) Evaluate the line integral

$$\int_C -y\,dx + x\,dy + z\,dz$$

where C is the helix $x = \cos t,\ y = \sin t,\ z = t$ for $0 \le t \le 2\pi$.

SKETCH OF SOLUTION. $2\pi + 2\pi^2$

123 (§16.2) Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y) = \langle xy, -y\rangle$ and $\vec{r}(t) = \langle t^4, t^5\rangle$ for $-1 \le t \le 1$.

SKETCH OF SOLUTION. $\displaystyle\int_{-1}^{1}(4t^{12} - 5t^9)\,dt = \frac{8}{13}$

124 (§16.2) Let $\vec{F}(x,y,z) = \langle y, x, z\rangle$ and let $\vec{r}(t) = \langle \cos(\pi t), \sin(\pi t), 1-t^2\rangle$, $-1 \le t \le 1$. Calculate $\int_C \vec{F}\cdot d\vec{r}$.

SKETCH OF SOLUTION. Because $\displaystyle\int_{-1}^{1} \sin^2 \pi t\, dt = \int_{-1}^{1} \cos^2 \pi t\, dt = 1$,
$\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} -2t(1-t^2)\, dt = 0.$

125 (§16.2) Let

$$\vec{G}(x,y,z) = x\vec{i} - z\vec{j} + y\vec{k}.$$

Calculate $\displaystyle\int_C \vec{G}\cdot d\vec{r}$, where C is the unit circle $y^2+z^2=1$ in the plane $x=0$ traversed in the counterclockwise fashion when viewed from the positive x-axis.

SKETCH OF SOLUTION. Parameterize circle $\vec{r}(t) = \langle 0, \cos t, \sin t\rangle, 0 \le t \le 2\pi$.
$\displaystyle\int_C \vec{G}\cdot d\vec{r} = \int_0^{2\pi} \langle 0, -\sin t, \cos t\rangle \cdot \langle 0, -\sin t, \cos t\rangle = 2\pi.$

126 (§16.3) Determine whether the vector fields (i) $\vec{G}(x,y,z) = x\vec{i} - z\vec{j} + y\vec{k}$ and (ii) $\vec{F}(x,y,z) = \langle y, x, z\rangle$ are conservative.

SKETCH OF SOLUTION. (i) Not conservative because $g_{yz} = -1$ and $g_{zy} = 1$.
(ii) Conservative: $f(x,y,z) = xy + z^2/2 + C$.

127 (§16.3) The vector fields (i) $\vec{F}(x,y,z) = \langle y, x, z\rangle$ and (ii) $\vec{H}(x,y) = \dfrac{x\vec{i} + y\vec{j}}{1+x^2+y^2}$ are conservative. For each, find a scalar potential.

SKETCH OF SOLUTION. (i) $f(x,y,z) = xy + z^2/2 + C$.
(ii) $h(x,y) = \frac{1}{2}\ln(1+x^2+y^2) + C$

128 (§16.3) Let $\vec{r}(t) = \langle \cos(\pi t), \sin(\pi t), 1-t^2\rangle$, $-1 \le t \le 1$. Use the fact that the vector field $\vec{F} = \langle y, x, z\rangle$ is conservative to calculate $\int_C \vec{F}\cdot d\vec{r}$.

SKETCH OF SOLUTION. Since C is a closed path, $\int_C \vec{F}\cdot d\vec{r} = 0$.

129 (§16.3) Let $\vec{F}(x,y) = P\vec{i} + Q\vec{j}$ denote the vector field

$$\vec{F}(x,y) = \frac{-y\vec{i} + x\vec{j}}{x^2+y^2}, \quad x^2+y^2 > 0.$$

(a) Let C denote the unit circle traversed counterclockwise. Argue from your sketch that

$$\int_C \vec{F}\cdot d\vec{r} > 0.$$

(b) Explain how (a) shows that $\vec{F}$ is not conservative (in its domain $x^2 + y^2 > 0$).

(c) Let $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ for $x > 0$. Show $\nabla f = \vec{F}$ for $x > 0$. Why does this, together with the observation above, not contradict the fundamental theorem of line integrals?

(d) Find $\int_C \vec{F} \cdot d\vec{r}$ where C is the square with vertices $(2, 0)$, $(2, 1)$, $(1, 1)$, $(1, 0)$ traversed counterclockwise.

Sketch of solution. (a) Since $\vec{r}(\theta) = \langle \cos\theta, \sin\theta \rangle, 0 \le \theta \le 2\pi, \vec{F} = \langle -\sin\theta, \cos\theta \rangle$ and $\vec{r}' = \langle -\sin\theta, \cos\theta \rangle$, $\vec{F} \cdot \vec{r}' > 0$. In fact, $\int_C \vec{F} \cdot d\vec{r} = 2\pi$. (b) If $\vec{F}$ were conservative, the line integral would be 0 for every closed curve C. (c) Not defined for $x = 0$ (d) 0 (conservative).

130 (§16.4) Use Green's Theorem to compute

$$\oint_C y^2 dx + x^2 dy,$$

where C is the boundary of the region determined by $y = x^2$ and $y = 1$, oriented counterclockwise.

Sketch of solution. $\displaystyle\int_{-1}^{1}\int_{x^2}^{1} (2x - 2y)\, dy\, dx = -\frac{8}{5}.$

131 (§16.4) Use Green's Theorem to find the area enclosed by $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = 1$. (Suggestion. Note that $x = \cos^4(t), y = \sin^4(t)$, $0 \le t \le \frac{\pi}{2}$ parameterizes part of the boundary curve.)

Sketch of solution. Let D be the region in the first quadrant bounded by three curves. Along $r_1 = \langle t, 0 \rangle, 0 \le t \le 1, \frac{1}{2}\int_{C_1} x\, dy - y\, dx = 0$. Along $r_2 = \langle \cos^4 t, \sin^4 t \rangle, 0 \le t \le \pi/2, \frac{1}{2}\int_{C_2} x\, dy - y\, dx = 1/6$. Along $r_3 = \langle 0, 1 - t \rangle, 0 \le t \le 1, \frac{1}{2}\int_{C_3} x\, dy - y\, dx = 0$. Total area $4(1/6) = 2/3$.

132 (§16.5) Let $\vec{F}(x, y, z) = \langle e^x + z, xy, z \rangle$.

(a) Compute the curl of $\vec{F}$.
(b) Is $\vec{F}$ conservative?
(c) What is the divergence of the curl of $\vec{F}$?

Sketch of solution. (a) curl$\vec{F} = \langle 0, 1, y \rangle$, (b) No, because curl$\vec{F} \ne \vec{0}$, (c) div(curl $\vec{F}$) $= 0$.

133 (§16.7) Let $\vec{G}(x,y,z) = \langle 0,1,y\rangle$. Let T denote the portion of the surface $z = y^3$ inside the cylinder $x^2 + y^2 = 1$ oriented with normal vector having positive z component. Compute,

$$\int\int_T \vec{G}\cdot\vec{n}\,dS.$$

SKETCH OF SOLUTION. $\int\int_T \vec{G}\cdot\vec{n}\,dS = \int\int_D \langle 0,1,y\rangle\cdot\langle 0,-3y^2,1\rangle\,dA = \int\int_D(-3y^2+y)\,dA = \int_0^{2\pi}\int_0^1 \left[-3(r\sin\theta)^2 + r\sin\theta\right] r\,dr\,d\theta = -\frac{3\pi}{4}$.

134 (§16.8) Let C denote the intersection of $z = y^3$ and $x^2 + y^2 = 1$ oriented counterclockwise when viewed from the point $(0,0,2)$. Let $\vec{F}(x,y,z) = \langle e^x + z, xy, z\rangle$. Use the results of problems 132 and 133 to find

$$\int_C \vec{F}\cdot d\vec{r}.$$

SKETCH OF SOLUTION. By Stokes's Theorem, you get the same answer as problem 2. As a line integral, use $\vec{r}(t) = \langle\cos t, \sin t, \sin^3 t\rangle, 0 \le t \le 2\pi$. $\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\langle e^{\cos t} + \sin^3 t, \cos t\sin t, \sin^3 t\rangle\cdot\langle -\sin t, \cos t, 3\sin^2 t\cos t\rangle\,dt = \int_0^{2\pi}\left[-\sin t\, e^{\cos t} - \sin^4 t + \cos^2 t\sin t + 3\sin^5 t\cos t\right]\,dt = -\frac{3\pi}{4}$.

135 (§16.7) Let S denote the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with outward pointing normal $\vec{n}$. Let $\vec{F}$ denote the vector field $\vec{F}(x,y,z) = \langle x,0,0\rangle$. Compute,

$$\int\int_S \vec{F}\cdot\vec{n}\,dS.$$

SKETCH OF SOLUTION. Parametrize the ellipsoid $\vec{r}(\phi,\theta) = \langle a\sin\phi\cos\theta, b\sin\phi\sin\theta, c\cos\phi\rangle, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$. $\vec{r}_\phi \times \vec{r}_\theta = \langle bc\sin^2\phi\cos\theta, ac\sin^2\phi\sin\theta, ab\cos\phi\sin\theta\rangle$.
$\int\int_S \vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^{\pi} abc\,\sin^3\phi\cos^2\theta\,d\phi\,d\theta = \frac{4}{3}\pi abc$.

136 (§16.6) Fix $0 < b < a$. The function $\vec{r} = \langle x,y,z\rangle$, where

$$\begin{aligned} x(u,v) &= (a + b\cos(v))\cos(u) \\ y(u,v) &= (a + b\cos(v))\sin(u) \\ z(u,v) &= b\sin(v) \end{aligned}$$

for $-\pi \le u, v \le \pi$, parameterizes a torus. Sketch the torus and compute its surface area.

Sketch of solution. $\vec{r}_u \times \vec{r}_v = (a+b\cos v)\langle b\cos u\cos v, b\sin u\cos v, b\sin v\rangle$. $|\vec{r}_u \times \vec{r}_v| = b(a+b\cos v)$. Surface area $= \int\int_D |\vec{r}_u \times \vec{r}_v| = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} b(a+b\cos v)\, dv\, du = 4ab\pi^2 = (2\pi b)(2\pi a)$ (Pascal's Theorem applies here).

137 (16.8) Let $F(x,y,z) = \langle u, v, w\rangle$ where

$$\begin{aligned} u(x,y,z) &= e^z \sin(z) + xy \\ v(x,y,z) &= -xz \\ w(x,y,z) &= e^{x^2+z^2} + \tan(xy). \end{aligned}$$

Let T denote the surface $x^2 + y^2 + z^4 = 1$, $z \geq 0$. Find

$$\int\int_T \nabla \times \vec{F} \cdot \vec{n}\, dS.$$

Sketch of solution. Use Stokes's Theorem: $\vec{r}(t) = \langle \cos\theta, \sin\theta, 0\rangle, 0 \leq \theta \leq 2\pi$. $\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \cos\theta\sin\theta, 0, e^{\cos^2\theta} + \tan(\cos\theta\sin\theta)\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle\, dt = \int_0^{2\pi} -\cos\theta\sin^2\theta\, d\theta = 0$.

138 (§16.9) Use the divergence theorem to find the volume of the ellipsoid E given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

(Suggestion: Consider problem 135.)

Sketch of solution. If $\vec{F} = \langle x, 0, 0\rangle$, then $\text{div}\vec{F} = 1$. Volume of ellipsoid E is $\int\int\int_E 1\, dV = \int\int\int_E \text{div}\vec{F}\, dV = \int\int_S \vec{F} \cdot dS = \frac{4}{3}\pi abc$.

139 (§16.8 (or 16.9)) Let $\vec{G}(x,y,z) = \langle 3y, -xz, yz^2\rangle$ and let $\vec{F} = \nabla \times \vec{G}$. Let T denote the union of the truncated paraboloid $z = \frac{1}{4}(x^2+y^2)$, $0 \leq z \leq 2$ with the truncated cylinder $x^2 + y^2 = 1$, $2 \leq z \leq 3$ with outward pointing normal. Compute

$$\int\int_T F \cdot \vec{n}\, dS.$$

Sketch of solution. The boundary at the top is given by $\vec{r}(t) = \langle \cos t, -\sin t, 3\rangle, 0 \leq t \leq 2\pi$. By Stokes's Theorem, $\int_C \vec{G} \cdot d\vec{r} = \int_0^{2\pi} \langle -3\sin t, -3\cos t, -9\sin t\rangle \cdot \langle -\sin t, -\cos t, 0\rangle\, dt = \int_0^{2\pi} [3\sin^2 t + 3\cos^2 t + 0]\, dt = 6\pi$.

140 (§16.8,16.9) Let $\vec{F}$ and $\vec{G}$ be as in the previous problem. Let S denote the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with outward pointing normal. Find

$$\int\int_S \vec{F} \cdot \vec{n}\, dS$$

two ways: first using Stokes' Theorem and second using the divergence theorem.

Sketch of solution. Answer is 0.