Notes for a talk about packing densities of permutations at the Graduate Student Combinatorics Seminar Rutgers University

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Permutations will be thought of as ways to order $\{1, 2, \ldots, n\}$ — no group theory involved.

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253 is of the same "type" as 132 because the smallest element comes first, the largest comes second, and the middle element comes last.

We write $253 \cong 132$.

Containment defined by example... 21453 contains 4 copies of 132:

243
253
143
153

Let $\sigma \in S_k$, $\pi \in S_n$. The **density of σ in π** is:

$$\begin{aligned} d(\sigma, \pi) &= \Pr(\text{a randomly chosen } k\text{-element subperm of } \pi \text{ is } \cong \sigma) \\ &= \frac{\#\text{ copies of } \sigma \text{ in } \pi}{\binom{n}{k}} \end{aligned}$$

Define

$$\delta_n(\sigma) = \max_{\pi \in S_n} d(\sigma, \pi).$$

This is the best we can do with an n-permutation.

We want to define the **packing density of σ** to be

$$\delta(\sigma) = \lim_{n \to \infty} \delta_n(\sigma),$$

but first we need to show that this limit exists.

δ_n is decreasing:

(1) Choose $\pi \in S_n$ so that $d(\sigma, \pi) = \delta_n(\sigma)$.

(2) Consider choosing a random k-element subperm of π by the following method:

 (a) Choose one element x to ignore.

 (b) Choose a k-element subperm from the rest, $\pi - x$, uniformly at random.

(3) This gives

$$\delta_n(\sigma) = d(\sigma, \pi) = \frac{1}{n} \sum_{x \in \pi} d(\sigma, \pi - x) \leq d(\sigma, \pi - \hat{x}),$$

for some specific $\hat{x} \in \pi$, and we have

$$d(\sigma, \pi - \hat{x}) \leq \delta_{n-1}(\sigma).$$

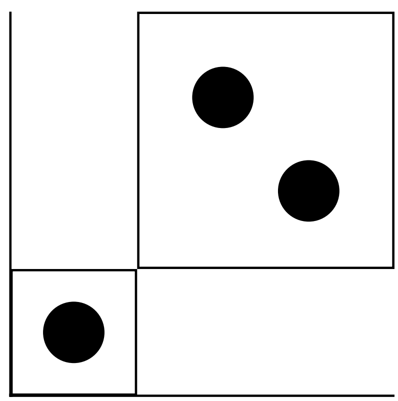
The first to ask about these questions was Herb Wilf, in 1992.

For example, $\delta(1) = \delta(12) = \delta(123) = \ldots = 1$.

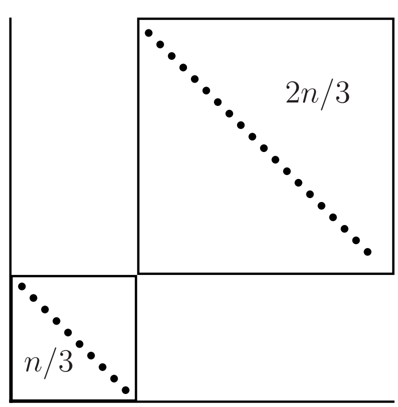
So out of S_3, we know $\delta(123)$ and $\delta(321)$, but we don't know the densities of 132, 213, 231, and 312.

By symmetry, it suffices to find $\delta(132)$.

Let's look at the plot of 132:



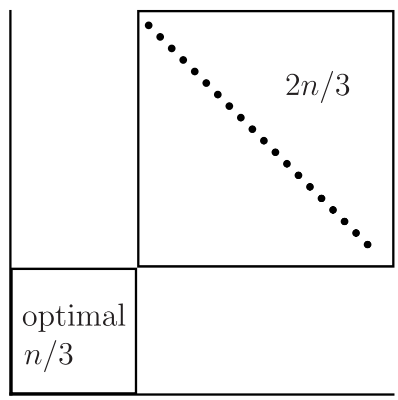
So now let's try to make a permutation that "looks like" 132.
Divide the permutation into 1/3 small elements and 2/3 big elements.



Using the probabilistic viewpoint, the density of 132 in this permutation is given by

$$\begin{aligned} d &= (\text{\# positions the small element could be in}) \cdot \Pr(\text{choose 1 small, 2 big}) \\ &= 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 = \frac{4}{9} = 0.444\ldots \end{aligned}$$

But we could do better... the small elements aren't being forced to do anything! Let's take our small elements to form a 132-optimal permutation.



The density here is given by

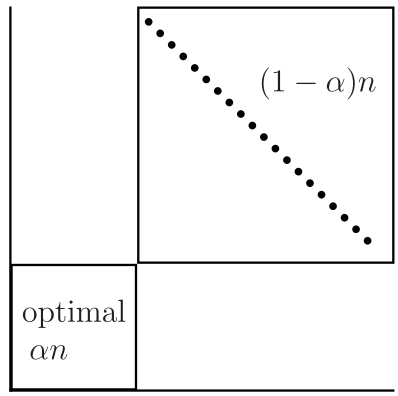
$$d = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^3 d$$

and solving this gives

$$d = \frac{6}{13} \approx 0.462.$$

But maybe the $\frac{1}{3} - \frac{2}{3}$ divider isn't optimal.

Let's have αn small elements and $(1-\alpha)n$ large elements, and have the αn small elements form an optimal permutation.



Then

$$d = 3\alpha(1-\alpha)^2 + \alpha^3 d,$$

and solving this gives

$$d = \frac{3(1-\alpha)\alpha}{1+\alpha+\alpha^2}$$

Optimizing for α gives

$$\alpha = \text{root of } 2x^3 - 3x + 1 = \frac{\sqrt{3}}{2} - \frac{1}{2} \approx 0.366$$

and

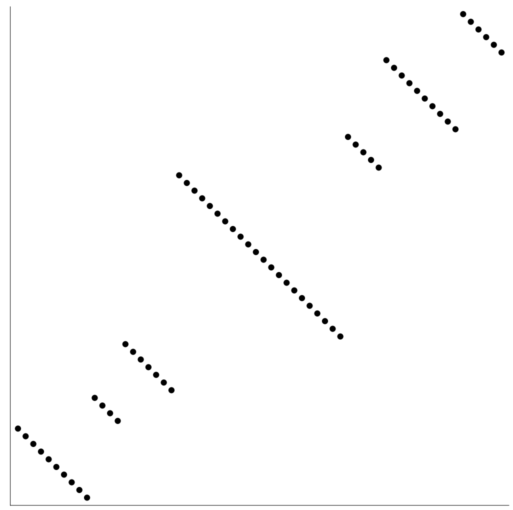
$$d = 2\sqrt{3} - 3 \approx 0.464.$$

This is indeed the packing density of 132, found independently by Galvin, Kleitmann, and Stromquist (all unpublished).

But of course our demonstration was not rigorous.

For one, what about optimal permutations with some other kind of structure?

Layered permutations defined by picture:



Stromquist's Theorem: Suppose that σ is a layered permutation. Then for every n there is a layered σ-optimal n-permutation.

Note: this says $\exists$, not $\forall$.

So our calculation of $\delta(132)$ was essentially correct.

Similarly,

$$\delta(1(\ell+1)\ell\ldots 2) = \ell\alpha(1-\alpha)^{\ell-1},$$

where

$$\alpha = \text{unique root of } \ell x^{\ell+1} - (\ell+1)x + 1 \text{ in } (0,1).$$

Here α is giving the asymptotic proportion of "small elements" in an optimal permutation.

This is due to **Price (1997)**.

ℓ	α	$\delta(1(\ell+1)\ell\ldots 2)$
2	0.366	0.464
3	0.253	0.424
4	0.200	0.410
5	0.167	0.402

Even with these permutations, the picture isn't entirely clear.

We know how many small elements an optimal permutation should have *asymptotically*, but what about for specific values of n?

Will it be the nearest integer to αn?

No!

Hildebrand-Sagan-Vatter (2004): For $\ell \geq 3$ we have

$$\alpha(n-\ell) - 1 \leq \#\text{ small elements} \leq \alpha(n-\ell) + 1.$$

For $\ell = 2$, the lower bound is still true, but the upper bound becomes $\alpha n + 1/2$. The upper bound above is conjectured to be true in this case.

Conjecture (HSV): The -1 in the lower bound can be removed to give

$$\#\text{ small elements} \geq \alpha(n-\ell).$$

HSV: For all $\ell \geq 3$:

$$\delta \frac{(n-\ell)^{\ell+1}}{(\ell+1)!} \leq \# \text{ copies of } 1(\ell+1)\ldots 2 \text{ in optimal } n\text{-perm} \leq \delta \frac{n^{\ell+1}}{(\ell+1)!}.$$

(For $\ell = 2$ the numerator in the upper bound must be increased to $(n+1)^{\ell+1}$, although again it is conjectured that this isn't necessary.)

Where does all this come out of? Why does one need to have so much detail about $1(\ell+1)\ldots 2$-optimal permutations?

Pattern frequency sequences... Bóna-Sagan-Vatter (2002).

	# of n-permutations with i copies of 132						
i	n=1	n=2	n=3	n=4	n=5	n=6	n=7
1	1	2	5	14	42	132	429
2			1	5	21	84	330
3				4	23	107	464
4				1	14	82	410
5					12	96	526
6					5	55	394
7					3	64	475
8						37	365
9						29	360
10						22	298
11						10	281
12						**0**	175
13						2	206
14							126
15							93
16							55
17							23
18							14
19							13
20							1
21							2

This chart has lots of interesting properties.

To start, focus on the first row, which gives the number of n-permutations with no copies of 132. The row reads 1,2,5,14,42,132,429... the Catalan numbers. (Proved by **Knuth (1973)**).

You could then move on to the next row, the number of permutations with 1 copy of 132. These numbers are 1,5,21,84,330.

Noonan and Zeilberger (1996) conjectured that this sequence is $\binom{2n-3}{n}$, and **Bóna (1998)** proved the conjecture.

You could keep doing this forever of course, 2 copies, 3 copies, etc. **Bóna (1997)** proved that these sequences are always P-recursive. **Mansour and Vainshtein (2002)** proved that the sequences have algebraic generating functions and gave an algorithm to compute them.

Instead of looking at the rows of this chart, let's look at the columns.

Observe that for $n = 1, 2, 3, 4, 5$, and 7, there are no "internal zeros" in the columns, but for $n = 6$, there is an internal zero, and it occurs when you look at permutations with one less than the maximum number of copies of 132, that is, when you look at nearly 132-optimal permutations.

So, how many columns will have internal zeros? How many won't have internal zeros?

Using the bounds on the number of small elements in a 132-optimal permutation given before (actually, much cruder bounds suffice), one can get

Bóna-Sagan-Vatter (2002): There are infinitely many columns of this table without internal zeros. Moreover, if a column has an internal zero, that zero occur either 1 or 2 spots from the bottom.

That still doesn't tell us how many columns have an internal zero, although it does tell us where to look: the nearly optimal permutations.

This is what lead to those tight bounds on the number of small elements and the maximum number of copies of 132 shown before.

And here is the result, obtained using those bounds and a Stromquistian Theorem for nearly optimal permutations.

BSV: There are infinitely many columns of this table with internal zeros.

Of course, one can ask the same questions about $1(\ell+1)\ell\ldots 2$. Here it is known that there are infinitely many columns with internal zeros, and it is conjectured by BSV that there are only finitely many columns without internal zeros.

Packing densities of other layered permutations

2143:

Let's consider a layered permutation with s layers.

Of course, these layers should all be the same size (this can be rigorized).

The density, d, of this permutation (with respect to 2143) is then given by

$$\begin{aligned} d &= \text{(choose 1st)} \cdot \text{(\# places to choose another in same block)} \cdot \\ &\quad \Pr(\text{choose 3rd element in different block}) \cdot \Pr(\text{choose 4th element in that block}) \\ &= 3 \cdot \left(\frac{1}{s}\right) \cdot \left(\frac{s-1}{s}\right) \cdot \left(\frac{1}{s}\right) \end{aligned}$$

This is maximized when $s = 2$, giving

$$\delta(2143) = \frac{3}{8} = 0.375.$$

First conjectured by Stromquist, proved by Price.

Note that the 2143-optimal permutations have a bounded number (2) of layers, while the 132-optimal permutations have an unbounded number of layers. How can you tell which behavior you'll get?

Albert-Atkinson-Handley-Holton-Stromquist (2002):

no length 1 layers	bounded
length 1 at beginning or end, or both	unbounded
layer lengths $a, 1, b$	bounded

Conjecture (AAHHS): If σs first and last layers have sizes greater than 1 and σ has no adjacent layers of length 1 then σ is of the bounded type.

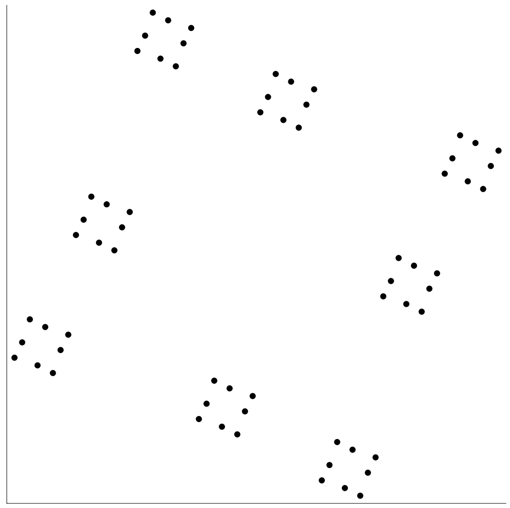
Now for all the known packing densities of *non*-layered permutations:

AAHHS: $0.0998 \approx \frac{51}{511} \leq \delta(2413) \leq \frac{2}{9} = 0.222\ldots$

The upper bound comes from the fact that the δ_ns are decreasing to δ.

For the lower bound, first note that 35827146 contains a good number (17) of copies of 2413.

So let's recursively structure our permutations like 35827146:



Now we can get a 2413 subperm in at least two different ways.

(1) By picking all a 2413 subperm that lies in a block. This has asymptotic probability

$$(\#\text{ of block}) \cdot \Pr(\text{picking all 4 in same block}) \cdot \Pr(\text{getting a 2413}) = 8\left(\frac{1}{8}\right)^4 d.$$

(2) By picking one element from each block of a 2413 subperm in 35827146. This has probability

$$(\#\text{ copies of 2413 in 35827146}) \cdot (\#\text{ different orders}) \cdot \Pr(\text{picking one from each block}) = 17 \cdot 24 \cdot \left(\frac{1}{8}\right)^4.$$

All of this gives

$$d \geq 8\left(\frac{1}{8}\right)^4 d + 17 \cdot 24 \cdot \left(\frac{1}{8}\right)^4,$$

from which the lower bound follows.