Maximal Independent Sets In Graphs With At Most r Cycles

Bruce E. Sagan
Department of Mathematics
Michigan State University
sagan@math.msu.edu

and

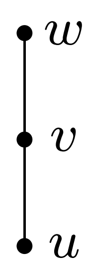
Vincent R. Vatter
Department of Mathematics
Rutgers University
vatter@math.rutgers.edu

Introduction

An independent set is *maximal* if it is contained in no larger independent set.

$m(G)$ = number of maximal independent sets in G

Ex. The path P_3



has maximal independent sets

$$\{u, w\} \text{ and } \{v\},$$

so $m(P_3) = 2$.

Given a family $\mathcal{F}$ of graphs with $|V| = n$ we ask two questions

1. What is $\max_{G \in \mathcal{F}} m(G)$?

2. What are the extremal graphs?

Erdős and Moser asked question 1 for the family of all graphs with n vertices in the early '60s. (Actually, they asked about maximal cliques.)

$\mathcal{F}$	Who	$When$	$Extremal$?
all graphs	Erdős	$\approx$ 1960	Y
all graphs	Moon and Moser	1965	Y
connected	Füredi	1987	N
connected	Griggs, Grinstead, and Guichard	1988	Y
trees	Wilf	1986	N
trees	Sagan	1988	Y
$\leq$ 1 cycle	Jou and Chang	1997	Y
$\leq$ 2 cycles	Goh and Koh	2001	Y
$\leq$ 3 cycles	Goh and Koh	2001	Y
$\leq r$ cycles	Sagan and Vatter	2001	Y

The family of all graphs

Let tG stand for the disjoint union of t copies of G.

Define

$$G(n) := \begin{cases} \frac{n}{3}K_3 & \text{if } n \equiv 0 \pmod 3, \\ 2K_2 \uplus \frac{n-4}{3}K_3 & \text{if } n \equiv 1 \pmod 3, \\ K_2 \uplus \frac{n-2}{3}K_3 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Also let

$$G'(n) := K_4 \uplus \frac{n-4}{3}K_3 \text{ if } n \equiv 1 \pmod 3.$$

Since $m(G \uplus H) = m(G)m(H)$,

$$g(n) := m(G(n)) = \begin{cases} 3^{\frac{n}{3}} & \text{if } n \equiv 0 \pmod 3, \\ 4 \cdot 3^{\frac{n-4}{3}} & \text{if } n \equiv 1 \pmod 3, \\ 2 \cdot 3^{\frac{n-2}{3}} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Note that $n \equiv 1 \pmod 3$ then $m(G'(n)) = m(G(n))$.

Theorem (Moon and Moser 1965) Let G be a graph with $n \geq 2$ vertices. Then

$$m(G) \leq g(n)$$

with equality if and only if $G \cong G(n)$ or, for $n \equiv 1 \pmod 3$, $G = G'(n)$.

The Main Theorem, all graphs case

Suppose that $n \geq 3r - 1$. Define

$$G(n,r) := \begin{cases} rK_3 \uplus \frac{n-3r}{2}K_2 & \text{if } n \equiv r \pmod 2, \\ (r-1)K_3 \uplus \frac{n-3r+3}{2}K_2 & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

Letting $g(n,r) = m(G(n,r))$,

$$g(n,r) = \begin{cases} 3^r \cdot 2^{\frac{n-3r}{2}} & \text{if } n \equiv r \pmod 2, \\ 3^{r-1} \cdot 2^{\frac{n-3r+3}{2}} & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

Theorem (Sagan and Vatter) Let G be a graph with n vertices and at most r cycles, where $r \geq 1$. If $n \geq 3r - 1$ then

$$m(G) \leq g(n,r)$$

with equality if and only if $G \cong G(n,r)$.

Recall...

A *block* in G is a maximal 2-connected subgraph of G.

An *endblock* is a block with at most one cutpoint.

The *block-cutpoint* graph of G is a graph that contains a vertex v_B for each block $B \subseteq G$ and a vertex v_x for each cutpoint $x \in G$.

Edges: $v_x v_B$ whenever $x \in V(B)$.

Proposition: Block-cutpoint graphs are forests.

Proposition: Every graph has an endblock that intersects at most one non-endblock.

Proof: Take a longest path in the block-cutpoint graph. Its two endpoints satisfy the desired condition.

Sketch of proof: Use double induction on n and r (the cases where $r \leq 3$ have been done previously). If $n = 3r - 1$ or $3r$ then

$$g(n, r) = g(n) \text{ and } G(n, r) = G(n)$$

so we are done by Moon-Moser.

One can show that if G has two or more intersecting cycles then $m(G) < g(n, r)$, so it suffices to look at graphs all of whose blocks are K_2 or cycles. Let B be an endblock of G. We have three cases depending on whether $B \cong K_2, K_3$, or C_p for $p \geq 4$.

The $B \cong K_2$ case: for any complete endblock B,

$$m(G) = \sum_{v \in V(B)} m(G - N[v])$$

where $N[v]$ is the closed neighborhood of v. Also, $g(n, r)$ is increasing in n. Let $V(B) = \{v, w\}$. Both $G - N[v]$ and $G - N[w]$ have at most $n - 2$ vertices and r cycles. By induction and the above facts

$$m(G) \leq 2g(n - 2, r) = g(n, r)$$

with equality iff $G - N[v] = G - N[w] \cong G(n - 2, r)$. It follows that B is actually a component of G isomorphic to K_2 so $G \cong G(n, r)$.

Are we really done?

Theorem (Sagan and Vatter) Let G be a graph with n vertices and at most r cycles, where $r \geq 1$. If $n \geq 3r - 1$ then

$$m(G) \leq g(n, r)$$

with equality if and only if $G \cong G(n, r)$.

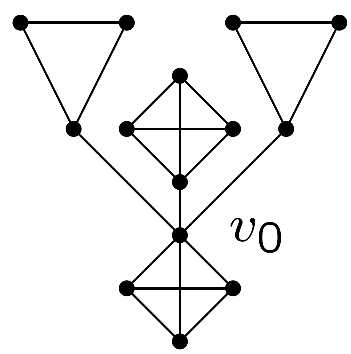
$$\#\text{ cycles in } G(n) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod 3, \\ \frac{n-4}{3} & \text{if } n \equiv 1 \pmod 3, \\ \frac{n-2}{3} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

If $n < 3r - 1$, the extremal graph is $G(n)$ (possibly $G'(n)$ too), and if $n \geq 3r - 1$ then the extremal graph is $G(n, r)$.

The family of all connected graphs

Construct $K_m * G$ by picking a vertex v_0 in K_m and joining it to one vertex in each component of G. For $n \geq 6$ let

$$C(n) := \begin{cases} K_3 * \frac{n-3}{3} K_3 & \text{if } n \equiv 0 \pmod 3, \\ K_4 * \frac{n-4}{3} K_3 & \text{if } n \equiv 1 \pmod 3, \\ K_4 * \left(K_4 \uplus \frac{n-8}{3} K_3 \right) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

$C(14) =$ 

Letting c(n):=m(C(n)):

$$c(n) = \begin{cases} 2 \cdot 3^{\frac{n-3}{3}} + 2^{\frac{n-3}{3}} & \text{if } n \equiv 0 \pmod 3, \\ 3^{\frac{n-1}{3}} + 2^{\frac{n-4}{3}} & \text{if } n \equiv 1 \pmod 3, \\ 4 \cdot 3^{\frac{n-5}{3}} + 3 \cdot 2^{\frac{n-8}{3}} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Theorem (Griggs, Grinstead, and Guichard) Let G be a connected graph with $n \geq 6$ vertices. Then

$$m(G) \leq c(n)$$

with equality if and only if $G \cong C(n)$.

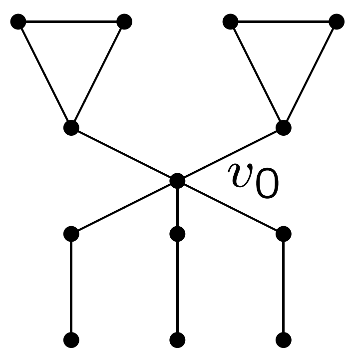
The Main Theorem, connected graphs case

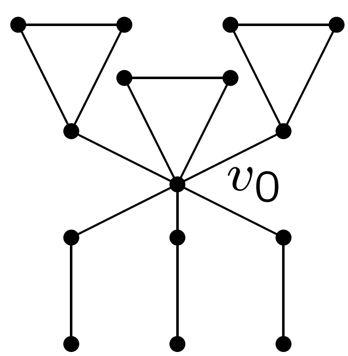
Suppose that $n \geq 3r$ and define

$$C(n,r) := \begin{cases} K_3 * \left((r-1)K_3 \uplus \frac{n-3r}{2}K_2\right) & \text{if } n \equiv r \pmod 2, \\ K_1 * \left(rK_3 \uplus \frac{n-3r-1}{2}K_2\right) & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

Letting $c(n,r) = m(C(n,r))$,

$$c(n,r) = \begin{cases} 3^{r-1} \cdot 2^{\frac{n-3r+2}{2}} + 2^{r-1} & \text{if } n \equiv r \pmod 2, \\ 3^r \cdot 2^{\frac{n-3r-1}{2}} & \text{if } n \not\equiv r \pmod 2. \end{cases}$$

$C(13,2) =$ 

$C(15,3) =$ 

The Main Theorem, connected graphs case

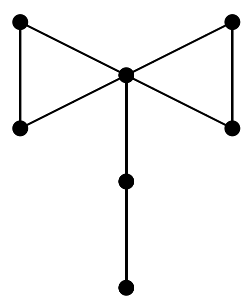
Theorem (Sagan and Vatter) Let G be a connected graph with n vertices and at most r cycles where $r \geq 1$. If $n \geq 3r$ then

$$m(G) \leq c(n, r)$$

with equality if and only if $G \cong C(n, r)$ or if G is one of the following exceptional cases

n	r	extremal $G \ncong C(n, r)$
4	1	P_4
5	1	C_5
7	2	$C(7, 1), E$

where

$E =$ 

(The proof is very similar to the other case.)

Are we really done?

(We had to assume $n \geq 3r$.)

Suppose $n \equiv 0 \pmod 3$. Since $C(n)$ has only $\frac{n-3}{3}$ cycles, it is the extremal graph when $r \geq \frac{n-3}{3}$, and the theorem applies when $r \leq n/3$.

But when $n \equiv 1 \pmod 3$,

$$C(n) = K_4 * \frac{n-4}{3} K_3$$

has $7 + \frac{n-4}{3} = \lfloor n/3 \rfloor + 6$ cycles. Our theorem only applies for $r \leq n/3$.

When $n \equiv 2 \pmod 3$,

$$C(n) = K_4 * \left(K_4 \uplus \frac{n-8}{3} K_3 \right)$$

has $\lfloor n/3 \rfloor + 12$ cycles, an ever wider gap!

Filling the gap

What goes wrong with the proof? Just the following statement:

"One can show that if G has two or more intersecting cycles then $m(G) < g(n, r)$, so it suffices to look at graphs all of whose blocks are K_2 or cycles."

Without this, we don't know what the endblocks look like.

Proposition: A graph B is 2-connected if and only if there is a sequence

$$B_0, B_1, \ldots, B_\ell = B$$

such that:

- B_0 is a cycle,
- B_{i+1} is obtained by taking a nontrivial path and identifying its two endpoints with two distinct vertices of B_i.

Every time we add a new path in this manner to a block, we increase the number of cycles in the block.

Lemma: (The $n \equiv 1 \pmod 3$ case) If there is a non-cutpoint v with $\deg v \geq 3$ lying on at least 6 cycles then G is not extremal.

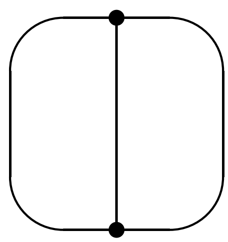
Lemma: (The $n \equiv 2 \pmod 3$ case) If non-cutpoint v with $\deg v \geq 3$ lying on at least 12 cycles then G is not extremal.

So, there are only finitely many different “types” of blocks to check.

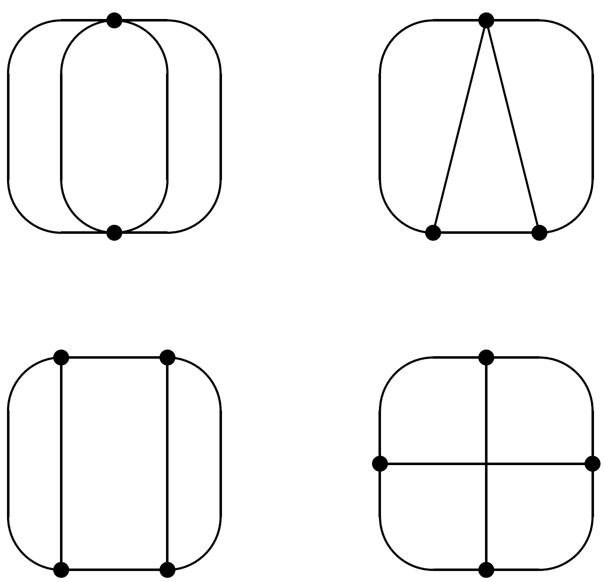
How many different types of blocks do we have to check?

If $\ell = 0$, then B is a cycle.

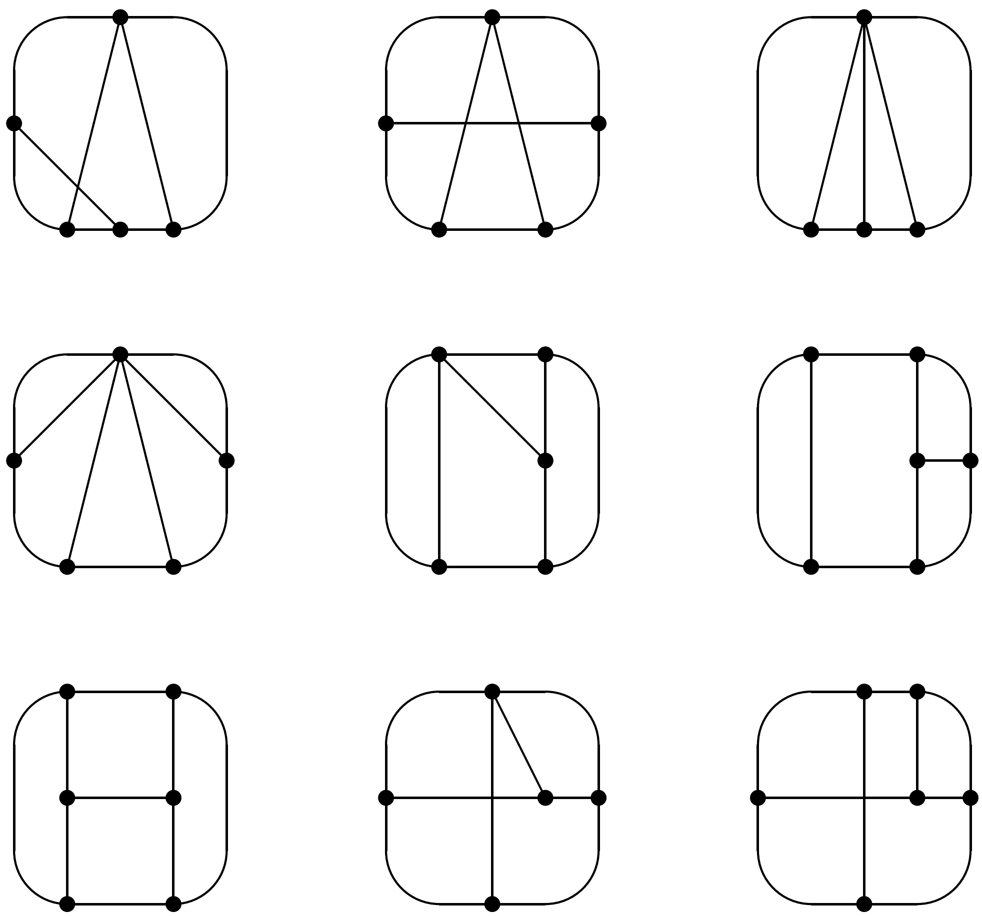
If $\ell = 1$, then B is a subdivision of



For $\ell = 2$ we have to check subdivisions of



Finally, if $\ell = 3$ we have to check subdivisions of



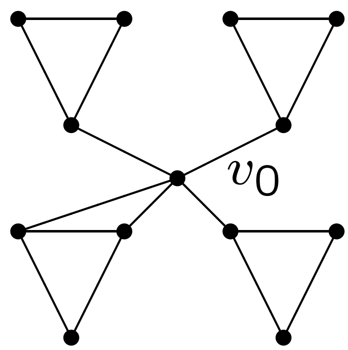
(No cases with $\ell = 4$ to check.)

The theorem we get

In the $n \equiv 1 \pmod 3$ case we find two new extremal graphs.

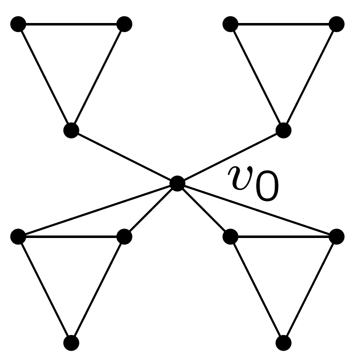
$C(n, \lfloor n/3 \rfloor + 2)$: Take a vertex v_0 and join it to one vertex in each of $\frac{n-4}{3}$ copies of K_3 and two vertices in another copy of K_3.

$C(13, 6) =$



And $C(n, \lfloor n/3 \rfloor + 4)$: Add another edge between v_0 and one of the $\frac{n-4}{3}$ copies of K_3 in the previous graph.

$C(13, 8) =$



Theorem (Sagan and Vatter) Let G be a connected graph with $n \equiv 1 \pmod 3$ vertices and at most r cycles, where $r \in [\lfloor n/3 \rfloor, \lfloor n/3 \rfloor + 5]$. Then

$$m(G) \le c(n, \lfloor n/3 \rfloor)$$

with equality if and only if
$G \cong C(n, \lfloor n/3 \rfloor), C(n, \lfloor n/3 \rfloor + 2)$, or $C(n, \lfloor n/3 \rfloor + 4)$.