Math 454 Summer 2005 Due Tuesday 8/2/05

Homework #4

First some review problems:

1. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in nonnegative integers not exceeding 8.

2. Determine a general formula for the number of permutations of the set $\{1, 2, 3, \ldots, n\}$ in which exactly k integers are in their natural positions. For example, in the permutation 1436527 the integers 1, 5, and 7 occur in their natural positions.

3. Let D_n denote the number of derangements of $\{1, 2, 3, \ldots, n\}$. We proved in class that

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{n!}\right).$$

Use this to establish recurrence relation $D_n = (n-1)(D_{n-2} + D_{n-1})$ (for $n \geq 3$).

⋆ 4. Give a combinatorial proof of this recurrence relation.

5. We proved in class that the nth Fibonacci number, f_n, is given by the formula

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Prove that f_n is actually the nearest integer to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

6. Solve the recurrence relation given by

$$\begin{aligned} a_n &= 2a_{n-1} + \binom{n}{2} \quad \text{for } n \geq 1, \\ a_0 &= 0. \end{aligned}$$

7. Solve the recurrence relation

$$a_n = 8a_{n-1} - 16a_{n-2},$$

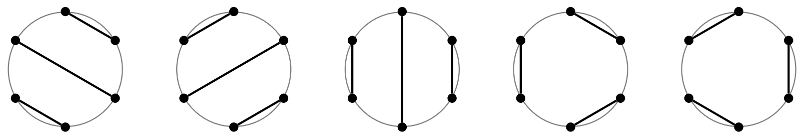
(for $n \geq 2$) with $a_0 = -1$ and $a_1 = 0$.

8. Solve the recurrence relation given by

$$\begin{aligned} a_n &= (n+2)a_{n-1} \quad \text{for } n \geq 1, \\ a_0 &= 2. \end{aligned}$$

9. Determine a recurrence relation for the number, a_n, of ternary strings (i.e., strings made up of 0's, 1's, and 2's) of length n that contain neither two consecutive 0's nor two consecutive 1's. Then use this recurrence relation to find a formula for a_n.

10 Fix $2n$ equally spaced points on a circle. Show that the number of ways to join these points in pairs so that the resulting n line segments do not intersect equals the n^{th} Catalan number. For example, here are the 5 ways to do this when $n = 3$:



11 Show that the number of permutations of $\{1, 2, 3, \ldots, n\}$ that do not have an *acb*-pattern is the n^{th} Catalan number[1].

12 Determine the generating function for the number, a_n, of bags contain n total apples, oranges, bananas, and pears in which:

(a) the number of apples is even,

(b) there are at most two oranges,

(c) the number of bananas is divisible by 3,

(d) there cannot be any pears if there are bananas, but otherwise there can be any number of pears.

13 Invent a combinatorial problem that gives the following generating function

$$(1 + x + x^2) \cdot \left(\frac{1}{1 - x^2}\right) \cdot \left(\frac{1}{1 - x^3}\right).$$

14 Complete the following 3×7 Latin rectangle

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 0 & 6 & 5 & 4 & 1 \\ 1 & 4 & 6 & 0 & 2 & 3 & 5 \end{bmatrix}.$$

15 How many $2 \times n$ Latin rectangles have first row equal to

$$\begin{bmatrix} 0 & 1 & 2 & \cdots & n-1 \end{bmatrix}?$$

[1] A permutation π has an *acb*-pattern if there are indices $1 \le i < j < k$ for which $\pi(i) < \pi(k) < \pi(j)$, i.e., their values read small, large, medium.