Math 454 Summer 2005 Due Tuesday 7/19/05

Homework #3

First some review problems:

1 Thirty toys are to be distributed among 10 children.

(a) How many ways are possible if the toys are all different?

(b) How many ways are possible if the toys are all identical and each child gets at least one toy?

(c) How many ways are possible if there are 6 different kinds of toys, with 25 of them identical and the other 5 different and special, and no child gets two special toys?

2 We showed in class that you can have 8 non-attacking rooks on a standard 8×8 chessboard, but you can't have any more. How many non-attacking kings[1] can you have?

Here are some problems using the binomial theorem. Very little explanation is required.

3 What is the coefficient of $x^{10}y^{97}$ in the expansion of

$$(3y - x)^{107}?$$

What is the coefficient of $x^{11}y^{95}$ in that expansion?

4 Evaluate the sum

$$\sum_{k=0}^{n} c^k \binom{n}{k},$$

where c is an arbitrary real number. Extra credit for a combinatorial proof in the case where c is a positive integer.

5 Evaluate the sum

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \left(\frac{1}{2}\right)^{n-k} \binom{n}{k}.$$

6 Evaluate the sum

$$\sum_{n=0}^{\infty} \frac{n}{\sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \binom{n}{k}}.$$

A multinomial theorem problem. No explanation required, just write down the answer (which you don't have to expand).

7 What is the coefficient of $x_1^9 x_2^1 x_4^6$ in the expansion of

$$(x_1 + x_2 + x_3 + x_4)^{16}?$$

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[1] Kings attack only adjacent squares, or in other words, they can move only square at a time, but in any direction.

Here is an identity that doesn't use the binomial theorem. I *am* looking for a nice explanation for this.

8 Give a combinatorial proof of the identity

$$\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}.$$

9 Using linear algebra, it's not hard to show that for every positive integer p, there are integers $c_1, c_2, \ldots, c_p$ so that

$$n^p = c_p\binom{n}{p} + c_{p-1}\binom{n}{p-1} + \cdots + c_2\binom{n}{2} + c_1\binom{n}{1}$$

for all positive integers n. (The ordering of these "for all"s is important!) For example,

$$n^2 = 2\binom{n}{2} + \binom{n}{1}$$

for all positive integers n. Important note: that's not the problem. Read below for the problem.

Find integers a, b, and c so that

$$n^3 = a\binom{n}{3} + b\binom{n}{2} + c\binom{n}{1},$$

and use this to sum the series $1^3 + 2^3 + 3^3 + \ldots + m^3$.

Here are some harder problems. Don't be worried if you can't solve them all.

⋆10 Terquem's problem: how many integer sequences $1 \le a_1 < a_2 < \cdots < a_k \le n$ are there so that $a_1, a_3, a_5, \ldots$ are odd and $a_2, a_4, a_6, \ldots$ are even?

⋆11 Define the numbers a_n by

$$\prod_{n=1}^{\infty}(1 + x^n) = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that a_n is the number of ways of writing n as the sum of distinct positive integers (where order doesn't matter). For example, the first few terms of this product are

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \cdots$$

and for these coefficients we have the following chart

n	a_n	ways to write n as the sum of distinct positive integers
0	1	the empty sum
1	1	1
2	1	2
3	2	$3 = 1 + 2$
4	2	$4 = 1 + 3$
5	3	$5 = 2 + 3 = 1 + 4$
6	4	$6 = 1 + 5 = 2 + 4 = 1 + 2 + 3$
7	5	$7 = 1 + 6 = 2 + 5 = 3 + 4 = 1 + 2 + 4$
8	6	$8 = 1 + 7 = 2 + 6 = 3 + 5 = 1 + 2 + 5 = 1 + 3 + 4$

⋆12 How many non-attacking knights[2] can you have on a standard chessboard?

[2] Knights move in the shape of an L – one square in one direction and two squares in a perpendicular direction – and they only capture pieces they land on, not pieces they jump over.