Math 454 Summer 2005 Due Wednesday 7/13/05

Homework #2

Counting problems:

1 How many permutations of $\{1,2,3,\ldots,12\}$ are there that don't begin with 2?

Solution: (100%) I think the easiest way is by subtracting off the bad permutations:

$$\begin{array}{rrcl} & 12! & = & \text{total number of permutations of } \{1,2,3,\ldots,12\} \\ - & 11! & = & \text{number of permutations of } \{1,2,3,\ldots,12\} \text{ that begin with 2} \\ \hline & 12!-11! & = & \text{number of permutations of } \{1,2,3,\ldots,12\} \text{ that don't begin with 2} \end{array}$$

You can also argue that there are 11 choices for the first element (not 2), 11 choices for the second element (not the first element), 10 for the third, and so on, and you'll get $11 \cdot 11!$, the same number. That works find on this problem, but not on the next.

2 How many permutations of $\{1,2,3,\ldots,12\}$ are there that don't begin with 2 *and* don't end with 9?

Solution: (70%) Again I think it's easiest to subtract (and for this problem, if you don't subtract, you have to be very careful):

$$\begin{array}{rrcl} & 12! & = & \text{total number of permutations of } \{1,2,3,\ldots,12\} \\ - & 10! & = & \text{number of permutations of } \{1,2,3,\ldots,12\} \\ & & & \text{that begin with 2 and end with 9} \\ \hline & 12!-11! & = & \text{number of permutations of } \{1,2,3,\ldots,12\} \\ & & & \text{that don't begin with 2 and don't end with 9} \end{array}$$

3 In how many ways can seven people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (but not necessarily on the same side)?

Solution: (60%) The number of circular permutations of 7 objects *is* $6!$, but this is *not* the answer. Suppose that the people are labeled $\{1,2,3,4,5,6,7\}$. The circular permutations are in bijection with (or, if you prefer, can be written as) permutations that start with 1, which is why there are only $6!$ of them. Now consider the two permutations 1234567 and 1765432. These specify the same arrangement of people, according to the rules of arrangements laid out in the problem. Indeed, 1432567 specifies the same arrangement of people as 1765234, and in general, the circular permutation $1, p_2, p_3, p_4, p_5, p_6, p_7$ specifies the same arrangement of people as the circular permutation $1, p_7, p_6, p_5, p_4, p_3, p_2, p_1$. So, to get the answer to the problem, we need to take the number of circular permutations of 7 people and divide by 2 to get $6!/2$.

4 How many 11-permutations of the multiset

$$\{3 \cdot a, 4 \cdot b, 5 \cdot c\}$$

are there?

Solution: (90%) Since there are 12 elements in this multiset, we can't just apply Theorem 3.4.2, which gives the number of permutations of a multiset, not the number of *partial* permutations of a multiset (unless we're clever, see below). Instead, not that an 11-permutation of this multiset will have to choose exactly one element to leave out, so we get

$$\begin{array}{rcl} \dfrac{11!}{2!4!5!} & = & \text{number of permutations that leave out an } a \\ \dfrac{11!}{3!3!5!} & = & \text{number of permutations that leave out a } b \\ + \qquad \dfrac{11!}{3!4!4!} & = & \text{number of permutations that leave out a } c \\ \hline \dfrac{11!}{2!4!5!}+\dfrac{11!}{3!3!5!}+\dfrac{11!}{3!4!4!} & = & \text{number of 11-permutations of } \{3\cdot a, 4\cdot b, 5\cdot c\} \end{array}$$

That's the answer. Note that if you simplify this expression, you get

$$\frac{11!}{2!4!5!}+\frac{11!}{3!3!5!}+\frac{11!}{3!4!4!}=\frac{12!}{3!4!5!},$$

the number of 12-permutations (or, just permutations) of our multiset. Why is this? Out of laziness, I'll just quite Marc: "each 11-permutation can be converted into a unique 12-permutation by adding the missing element to the end and each 12-permutation can be converted to a unique 11-permutation by chopping off the last number."

5 List all 3-combinations and 4-combinations of the multiset $\{2\cdot a, 1\cdot b, 3\cdot c\}$.

Solution: (100%) First note that our formula for r-combinations of a multiset (Theorem 3.5.1) only works when we have infinite repetition. Now, 3-combinations:

$$\{a,a,b\},\{a,a,c\},\{a,b,c\},\{a,c,c\},\{b,c,c\},\{c,c,c\},$$

and 4-combinations:

$$\{a,a,b,c\},\{a,a,c,c\},\{a,b,c,c\},\{a,c,c,c\},\{b,c,c,c\}.$$

6 A bakery sells 6 different kinds of pastry. If the bakery has virtually unlimited supply of each kind, how many different options for a dozen of pastry are there? What if a box is to contain at least one of each kind of pastry?

Solution: (70%) For the first part, the answer is the number of integral solutions to

$$x_1+x_2+x_3+x_4+x_5+x_6=12,$$

with $x_1,x_2,x_3,x_4,x_5,x_6\geq 0$, which is $\binom{12+6-1}{6}=\binom{17}{6}$. The answer to the second part is the number of integral solutions to

$$x_1+x_2+x_3+x_4+x_5+x_6=12,$$

where $x_1, x_2, x_3, x_4, x_5, x_6 \geq 1$. You use "balls and walls" to get the answer (for that matter, you could use balls and walls on the first part); there are 5 walls (dividing the variables) and 11 spots for them (since no two walls can occupy the same spot), and this gives $\binom{11}{5}$. You could also set $y_i = x_i - 1$, so now we want to count integral solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 6$$

with $y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$. The number of these is $\binom{6+6-1}{6} = \binom{11}{6}$.

7 How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

are there that satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, and $x_4 \geq 8$?

Solution: (100%) I would start by making the substitutions

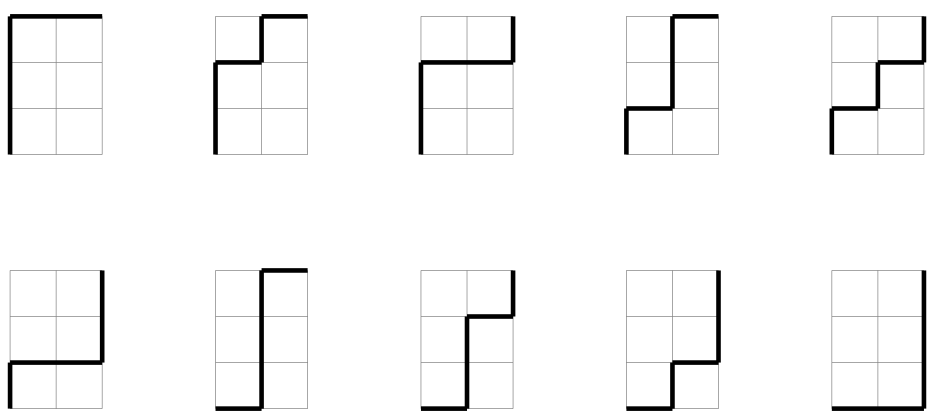
$$\begin{aligned} y_1 &= x_1 - 2, \\ y_2 &= x_2, \\ y_3 &= x_3 + 5, \\ y_4 &= x_4 - 8, \end{aligned}$$

so now we want to count integral solutions to

$$y_1 + y_2 + y_3 + y_4 = 25$$

that satisfy $y_1, y_2, y_3, y_4 \geq 0$. There are $\binom{25+4-1}{25} = \binom{28}{25}$ of these.

8 How many paths are there from the point $(0, 0)$ to the point (m, n) if each step is either $(1, 0)$ or $(0, 1)$, or in other words if each step is either one unit east or one unit north? For example, the 10 paths of this type from $(0, 0)$ to $(2, 3)$ are shown below.



Solution: (60%) To get from $(0, 0)$ to (m, n) with only east and north steps we will have to take $m + n$ total steps, m of them east and n north. So the number of paths is

$$\binom{m+n}{m} = \binom{m+n}{n}.$$

⋆ 9 Prove that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

Solution: (60%) **Proof #1:** First, props to Kate for finding the quick solution, which goes like this. The binomial theorem says

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Now differentiate this once to get

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^k,$$

and plug in $x = 1$ to get

$$n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k},$$

which is what we wanted.

Proof #2: Mike gave a proof using the fact that $k\binom{n}{k} = n\binom{n-1}{k-1}$, which can be verified by expressing $\binom{n}{k}$ and $\binom{n-1}{k-1}$ in terms of factorials. Using this identity, we have that

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n} n\binom{n-1}{k-1} = n\sum_{k=0}^{n} \binom{n-1}{k-1}. \tag{1}$$

Now we have to play around with $\sum_{k=0}^{n} \binom{n-1}{k-1}$ a little. The summand is 0 when $k = 0$, so we can write it as

$$\sum_{k=1}^{n} \binom{n-1}{k-1}.$$

Now substitute $j = k - 1$ to get

$$\sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}.$$

To finish, we just have to plug this in to (1) and get

$$\sum_{k=0}^{n} k\binom{n}{k} = n\sum_{j=0}^{n-1} \binom{n-1}{j} = n2^{n-1}.$$

Proof #3: Chris gave the combinatorial proof the hints hinted at, and phrased it better than the hints, so I'm going to copy his presentation here. Actually, I'm just going to copy his first two sentences. The left-hand side – $\sum_{k=0}^{n} k\binom{n}{k}$ – counts the number of possible selections of a committee of size k and the committee's president given n total people. The right-hand side can be thought of as counting the number of possible selections of a president and the rest of his/her committee members.

Proof #4: Marc gave the probabilistic proof, which goes as follows. First divide both sides by 2^n to get

$$\frac{\sum_{k=0}^{n} \binom{n}{k}}{2^n} = \frac{n}{2}.$$

Now both sides compute the average size of a subset. That $n/2$ is the average size of a subset follows from the fact that each element has a $1/2$ chance of being in any one subset, and there are n elements, so we multiply these two numbers together using a fancy principle called *linearity of expectation* to get an average of $n/2$.

Now we need to show that the left-hand side also computes the average size of a subset. To compute the average cardinality of a subset of $\{1, 2, 3, \ldots, n\}$ you would use the formula

$$\frac{\sum_{\text{subsets } A \subseteq \{1,2,3,\ldots,n\}} |A|}{\#\text{ of subsets of } \{1,2,3,\ldots,n\}}. \qquad (2)$$

When evaluating this formula, you will add k as many times as there is a k-element subset of $\{1, 2, 3, \ldots, n\}$, or in other words, $\binom{n}{k}$ times. So, we can rewrite (2) as

$$\frac{\sum_{k=0}^{n} \binom{n}{k}}{2^n},$$

proving the identity.

⋆10 Construct a cycle containing all the 2-combinations and 3-combinations of $\{1, 2, 3, 4, 5\}$ where each edge corresponds to either adding an element or removing an element.

Solution: (30%) Here's Mike's cycle:

$\{1,2\}$ — $\{1,2,3\}$ — $\{1,3\}$ — $\{1,3,5\}$ — $\{1,5\}$
$\{1,2\}$	$\{1,5\}$
$\{1,2,4\}$	$\{1,2,5\}$
$\{2,4\}$	$\{2,5\}$
$\{2,3,4\}$	$\{2,4,5\}$
$\{2,3\}$	$\{4,5\}$
$\{2,3,5\}$	$\{1,4,5\}$
$\{3,5\}$	$\{1,4\}$

$\{3,5\}$ — $\{3,4,5\}$ — $\{3,4\}$ — $\{1,3,4\}$ — $\{1,4\}$

Kate's cycle:

$\{1,2\}$	$\{1,2,4\}$	$\{2,4\}$	$\{2,3,4\}$	$\{2,3\}$
$\{1,2,5\}$				$\{1,2,3\}$
$\{1,5\}$				$\{1,3\}$
$\{1,4,5\}$				$\{1,3,5\}$
$\{1,4\}$				$\{3,5\}$
$\{1,3,4\}$				$\{2,3,5\}$
$\{3,4\}$	$\{3,4,5\}$	$\{4,5\}$	$\{2,4,5\}$	$\{2,5\}$

Marc's cycle:

$\{1,2\}$	$\{1,2,3\}$	$\{1,3\}$	$\{1,3,4\}$	$\{1,4\}$
$\{1,2,4\}$				$\{1,4,5\}$
$\{2,4\}$				$\{4,5\}$
$\{2,4,5\}$				$\{3,4,5\}$
$\{2,5\}$				$\{3,4\}$
$\{1,2,5\}$				$\{2,3,4\}$
$\{1,5\}$	$\{1,3,5\}$	$\{3,5\}$	$\{2,3,5\}$	$\{2,3\}$