Math 454 Summer 2005 Due Wednesday 7/6/05

Homework #1

The **pigeonhole principle** states that if $n+1$ objects are placed into n boxes then at least one box contains at least two elements. Use the pigeonhole principle to solve the following problems.

1. Prove that if $n+1$ numbers are chosen from the set $\{1, 2, 3, 4, \ldots, 2n\}$ then there must be two which differ by 1.

2. Prove that of any five points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/2$.[1]

3. Prove that if $n \geq 2$ people are at a party, then there are two people with the same number of acquaintances at the party. (Assume that no one is his or her own acquaintance.)

4. Judging by the last three drafts, it seems that the Detroit Lions want to draft at least one wide receiver a year for 13 years, ending up with exactly 20 wide receiver draftees. Prove that there will be a set of consecutive years in which the Lions draft exactly 5 wide receivers.

5. Consider the previous problem again. Will there be a set of consecutive years in which the Lions draft exactly 4 wide receivers? Can you prove that there will be more than one such set?

The **Erdős-Szekeres theorem** states that every sequence of at least n^2+1 real numbers contains a monotone (in other words, either non-decreasing or non-increasing) subsequence of length $n+1$.

6. The Erdős-Szekeres theorem says that a sequence containing at least 10 numbers must have a monotone subsequence of length 4.[2] Construct a sequence of 9 numbers that does *not* have a monotone subsequence of length 4.

7. How many length 9 permutations are there that don't have a monotone subsequence of length 4?

Here are two more questions that relate to the material in Chapter 2.

8. Show that $R(3, 3, 3) \leq 17$.

9. A collection of subsets of $\{1, 2, 3, \ldots, n\}$ has the property that every pair of subsets from this family intersects (has at least one element in common). Prove that there are at most 2^{n-1} subsets in the collection, and construct an example to show that there is such a collection containing 2^{n-1} subsets.[3]

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[1] Hint: the "boxes" you need for the pigeonhole principle are actually triangles.

[2] An aside: suppose that the sequence is of length $m \geq 10$. The Erdős-Szekeres theorem only guarantees one such subsequence. When m is much larger than 10 we should expect to find many monotone subsequences of length 4, but it remains an open problem to find the minimum number of length 4 monotone subsequences that a sequence of m numbers must have, as a function of m. See J. Myers, "The minimum number of monotone subsequences," *Electronic Journal of Combinatorics* **9 (2)** (2004), #R4, available electronically at `http://www.combinatorics.org/Volume_9/Abstracts/v9i2r4.html`.

[3] Hint: there are 2^n subsets of $\{1, 2, \ldots, n\}$, so 2^{n-1} is exactly half of this.

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The following two enumerative questions concern the material in Chapter 3.

10 Suppose that there are four women and six men and that each women marries one of the men. In how many ways can this be done?

11 How many subsets of $\{1, 2, \ldots, 8\}$ are there that contain at least one odd number?

Answers to the following questions will help me with the direction of the course, in addition to satisfying my curiosity.

12 Please describe your history with mathematics. Possible angles include: What classes have you enjoyed the most? What problems have interested you? How did you end up studying mathematics?

13 What are your future plans for mathematics? What are you studying here?