Math 454 Summer 2004

Exam #1

Thursday, 7/20/2005

Name: ______________________________

Problem	Points
1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
9	/10
10	/10
11	/20
12	/20
13	/20
14	/20
15	/20
Total	/200

1 (10 points) How many orderings of a deck of 52 cards are there if all the cards of the same suit are together? (A deck of cards has 4 suits, and each suit has 13 cards.)

2 (10 points) Evaluate the sum $\displaystyle\sum_{k=0}^{n}(-1)^{n-k}2^k\binom{n}{k}x^{n-k}$.

3 (10 points) How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 123$$

are there that satisfy $x_1 \geq 1$, $x_2 \geq -5$, $x_3 \geq 10$, and $x_4 \geq 56$?

4 (10 points) How many sequences $(a_1, a_2, \ldots, a_{13})$ are there consisting of four 0's and nine 1's if no two consecutive terms are both 0's?

5 (10 points) What is the coefficient of $w^8x^3yz^4$ in $(2w - x + 3y + z)^{16}$?

6 (10 points) Show that $\binom{n}{9} > \binom{n}{8}$ for all $n \geq 100$.

7 (10 points) Give a general formula for the number of r-combinations of the multiset $\{2 \cdot a, \infty \cdot b, \infty \cdot c\}$.

8 (10 points) How many walks are there from the point $(0, 0)$ to the point $(100, 100)$ using only right steps $(1, 0)$ and up steps $(0, 1)$?

9 (10 points) How many of the walks from the previous problem pass through both points $(9, 16)$ and $(12, 29)$?

10 (10 points) Construct the binary reflected Gray codes of length 1, 2, and 3.

11 (20 points) Prove the identity $\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$ (in any way you want).

12 Two parts:

(a) (5 points) Use the binomial theorem to expand $(1-x)^n$.

(b) (15 points) Evaluate the sum

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k} = 1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots \pm \frac{1}{n+1}\binom{n}{n}.$$

13 Two parts:

(a) (15 points) Show that no matter how you 2-color the edges of the complete graph on 6 vertices, K_6, there will be a monochromatic K_3.

(b) (5 points) Construct a 2-coloring of the edges of K_5 that does not contain a monochromatic K_3.

14 (20 points) Prove that no matter how ten points are chosen within an equilateral triangle of side length 1, there will be two whose distance apart is at most $1/3$.

15 (20 points) From May 15, 1941 to July 16, 1941, Joe DiMaggio had at least one hit in 56 straight games, and 91 hits overall[1]. Prove that there was a set of consecutive games during which DiMaggio had a total of exactly 20 hits.

[1] Stephen Jay Gould has declared DiMaggio's streak to be the greatest accomplishment in the history of baseball, in the sense that it is the least likely to ever happen again.