

**Points:** The first problem is worth 10 points, the others are worth 15.

1 Find the dual of the following linear programming problem.

$$\begin{aligned} &\text{Maximize } z = x - y \\ &\text{subject to} \\ &3x - y \leq 19 \\ &-x + 7y \geq 10 \\ &x + y = 100 \\ &x, y \geq 0 \end{aligned}$$

**Answer:** First I would convert the problem to standard form:

$$\begin{aligned} &\text{Maximize } z = x - y \\ &\text{subject to} \\ &3x - y \leq 19 \\ &x - 7y \leq -10 \\ &x + y \leq 100 \\ &-x - y \leq -100 \\ &x, y \geq 0 \end{aligned}$$

Then the dual problem is

$$\begin{aligned} &\text{Minimize } z' = 19w_1 - 10w_2 + 100w_3 - 100w_4 \\ &\text{subject to} \\ &3w_1 + w_2 + w_3 - w_4 \geq 1 \\ &-w_1 - 7w_2 + w_3 - w_4 \geq -1 \\ &w_1, w_2, w_3, w_4 \geq 0 \end{aligned}$$

2 Use the Dual Simplex Method to restore feasibility to the following tableau:

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$y$	0	1	$\frac{3}{10}$	$-\frac{1}{10}$	0	$\frac{24}{5}$
$x$	1	0	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{36}{5}$
$u_3$	0	0	$\frac{1}{5}$	$-\frac{3}{5}$	1	$-\frac{1}{5}$
	0	0	$\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{132}{5}$

**Answer:** The departing variable is  $u_3$ , the entering variable is  $u_2$ , and after pivoting we get

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$y$	0	1	$\frac{4}{15}$	0	$-\frac{1}{6}$	$\frac{29}{6}$
$x$	1	0	$-\frac{3}{5}$	0	1	7
$u_2$	0	0	$-\frac{1}{3}$	1	$-\frac{5}{3}$	$\frac{1}{3}$
	0	0	$\frac{7}{15}$	0	$\frac{1}{3}$	$\frac{79}{3}$

3 Consider the following linear programming problem.

$$\begin{aligned} &\text{Maximize } z = 4x_1 + 6x_2 + 2x_3 \\ &\text{subject to} \\ &x_1 + x_2 + x_3 \leq 10 \\ &x_1 + 4x_2 \leq 15 \\ &x_1 + x_3 \leq 6 \\ &x_1, x_2, x_3 \geq 0 \end{aligned}$$

Applying the Simplex Method to this problem yields the following final tableau

$\underline{c}_B$		4	6	2	0	0	0	
		$x_1$	$x_2$	$x_3$	$u_1$	$u_2$	$u_3$	
0	$u_1$	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
6	$x_2$	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
4	$x_1$	1	0	1	0	0	1	6
		0	0	$\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{75}{2}$

Suppose now that the problem is changed to “Maximize  $z = 3x_1 + 6x_2 + 2x_3$ .” Find an optimal solution to this new problem. (No points will be given for starting from scratch!)

**Answer:** First we need to update the tableau as usual to get

	$x_1$	$x_2$	$x_3$	$u_1$	$u_2$	$u_3$	
$u_1$	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
$x_2$	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
$x_1$	1	0	1	0	0	1	6
	0	0	$-\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{63}{2}$

After one iteration of the Simplex Method (with entering variable  $x_3$  and departing variable  $x_1$ ), we get the final tableau shown below.

	$x_1$	$x_2$	$x_3$	$u_1$	$u_2$	$u_3$	
$u_1$	$-\frac{1}{4}$	0	0	1	$-\frac{1}{4}$	-1	$\frac{1}{4}$
$x_2$	$\frac{1}{4}$	1	0	0	$\frac{1}{4}$	0	$\frac{15}{4}$
$x_3$	1	0	1	0	0	1	6
	$\frac{1}{2}$	0	0	0	$\frac{3}{2}$	2	$\frac{69}{2}$

4 Consider the following integer programming problem

$$\begin{aligned} &\text{Maximize } z = x + y \\ &\text{subject to} \\ &13x + 5y \leq 78 \\ &-x + y \leq 0 \\ &x, y, \geq 0, \text{ integral} \end{aligned}$$

Solve this problem using the Cutting Plane Method. (No points will be given for any other methods!)

To help you out, here is the final tableau for the corresponding non-integer linear programming problem (that is, if we ignore the word “integral”):

	$x$	$y$	$u_1$	$u_2$	
$x$	1	0	$\frac{1}{18}$	$-\frac{5}{18}$	$\frac{13}{3}$
$y$	0	1	$\frac{1}{18}$	$\frac{13}{18}$	$\frac{13}{3}$
	0	0	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{26}{3}$

**Answer:** We could add cutting planes to deal with either of the two constraints, and at this point we can't tell which one would be better to add, so I will choose the first constraint,

$$x + \frac{1}{18}u_1 - \frac{5}{18}u_2 = \frac{13}{3}.$$

(It turns out that adding a cutting plane to deal with the second constraint is a little less work.) Since  $x$ ,  $u_1$ , and  $u_2$  must all be non-negative, we can round down their coefficients to get a smaller left-hand side:

$$x - u_2 \leq \frac{13}{3}.$$

Now we can round down the right-hand side because  $x - u_2$  must be an integer:

$$x - u_2 \leq 4.$$

Finally, we add a new slack variable:

$$x - u_2 + u_3 = 4,$$

and place this constraint into our tableau:

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$x$	1	0	$\frac{1}{18}$	$-\frac{5}{18}$	0	$\frac{13}{3}$
$y$	0	1	$\frac{1}{18}$	$\frac{13}{18}$	0	$\frac{13}{3}$
$u_3$	1	0	0	-1	1	4
	0	0	$\frac{1}{9}$	$\frac{4}{9}$	0	$\frac{26}{3}$

But before applying the Dual Simplex Method we have to clean up  $x$ 's column:

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$x$	1	0	$\frac{1}{18}$	$-\frac{5}{18}$	0	$\frac{13}{3}$
$y$	0	1	$\frac{1}{18}$	$\frac{13}{18}$	0	$\frac{13}{3}$
$u_3$	0	0	$-\frac{1}{18}$	$-\frac{13}{18}$	1	$-\frac{1}{3}$
	0	0	$\frac{1}{9}$	$\frac{4}{9}$	0	$\frac{26}{3}$

Now we can apply the Dual Simplex Method. The departing variable will be  $u_3$ . The ratio for  $u_1$  is  $-2$  and the ratio for  $u_2$  is  $-8/13$ , so  $u_2$  will be the entering variable. After pivoting we have the following tableau

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$x$	1	0	$\frac{1}{13}$	0	$-\frac{5}{13}$	$\frac{58}{13}$
$y$	0	1	0	0	1	4
$u_2$	0	0	$\frac{1}{13}$	1	$-\frac{18}{13}$	$\frac{6}{13}$
	0	0	$\frac{1}{13}$	0	$\frac{8}{13}$	$\frac{110}{13}$

Unfortunately,  $x$  still isn't an integer! So, we have to add another cutting plane for the first constraint,

$$x + \frac{1}{13}u_1 - \frac{5}{13}u_3 = \frac{58}{13},$$

which becomes

$$x - u_3 + u_4 = 4$$

after rounding everything down and adding a new slack variable.

Thus our tableau is now

	$x$	$y$	$u_1$	$u_2$	$u_3$	$u_4$	
$x$	1	0	$\frac{1}{13}$	0	$-\frac{5}{13}$	0	$\frac{58}{13}$
$y$	0	1	0	0	1	0	4
$u_2$	0	0	$\frac{1}{13}$	1	$-\frac{18}{13}$	0	$\frac{6}{13}$
$u_3$	1	0	0	0	-1	1	4
	0	0	$\frac{1}{13}$	0	$\frac{8}{13}$	0	$\frac{110}{13}$

And after cleaning up the  $x$  column we get

	$x$	$y$	$u_1$	$u_2$	$u_3$	$u_4$	
$x$	1	0	$\frac{1}{13}$	0	$-\frac{5}{13}$	0	$\frac{58}{13}$
$y$	0	1	0	0	1	0	4
$u_2$	0	0	$\frac{1}{13}$	1	$-\frac{18}{13}$	0	$\frac{6}{13}$
$u_3$	0	0	$-\frac{1}{13}$	0	$-\frac{8}{13}$	0	$-\frac{6}{13}$
	0	0	$\frac{1}{13}$	0	$\frac{8}{13}$	0	$\frac{110}{13}$

Now we need to apply the Dual Simplex Method. The departing variable is  $u_3$  and the ratios for  $u_1$  and  $u_3$  are the same, so technically it doesn't matter which one we take as our entering variable. Except that it actually matters quite a bit, because if we take  $u_1$  as our entering variable we get

the following tableau:

	$x$	$y$	$u_1$	$u_2$	$u_3$	$u_4$	
$x$	1	0	0	0	-1	0	4
$y$	0	1	0	0	1	0	4
$u_2$	0	0	0	1	-2	0	0
$u_1$	0	0	1	0	8	0	6
	0	0	0	0	0	0	8

Since this tableau represents the integral solution  $x = 4, y = 4, z = 8$ , we are done. In case we take  $u_3$  as our entering variable, we get the following tableau:

	$x$	$y$	$u_1$	$u_2$	$u_3$	$u_4$	
$x$	1	0	$\frac{1}{8}$	0	0	0	$\frac{19}{4}$
$y$	0	1	$-\frac{1}{8}$	0	0	0	$\frac{13}{4}$
$u_2$	0	0	$\frac{1}{4}$	1	0	0	$\frac{3}{2}$
$u_1$	0	0	$\frac{1}{8}$	0	1	0	$\frac{3}{4}$
	0	0	0	0	0	0	8

And thus we have to keep adding cutting planes.

(Also, if we had just made the first cutting plane take care of the second constraint, we would have been done almost immediately.)

5 Solve the following integer programming problem using the Branch and Bound Method. (No points will be given for any other method!)

$$\begin{aligned} &\text{Maximize } z = 2x + 3y \\ &\text{subject to} \\ &2x - y \leq 0 \\ &2y \leq 21 \\ &x, y, \geq 0, \text{ integral} \end{aligned}$$

To help you out, here is the final tableau for the corresponding non-integer linear programming problem (that is, if we ignore the word “integral”):

	$x$	$y$	$u_1$	$u_2$	
$x$	1	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{21}{4}$
$y$	0	1	0	$\frac{1}{2}$	$\frac{21}{2}$
	0	0	1	2	42

**Answer:** Since our heuristic for choosing which variable to be our branch variable says to take the one that is furthest from it’s floor, we choose  $y$ . Then our two branches are  $y \leq 10$  and  $y \geq 11$ . To eliminate the  $y \geq 11$  branch we just need to notice that one of our constraints says  $2y \leq 21$ , so there are no feasible solutions with  $y \geq 11$ .

That leaves us with the  $y \leq 10$  node. So we add a new slack variable to get

$$y + u_3 = 10,$$

and add this constraint to our tableau, giving

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$x$	1	0	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{21}{4}$
$y$	0	1	0	$\frac{1}{2}$	0	$\frac{21}{2}$
$u_3$	0	1	0	0	1	10
	0	0	1	2	0	42

Of course we have to clean up the  $y$  column:

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$x$	1	0	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{21}{4}$
$y$	0	1	0	$\frac{1}{2}$	0	$\frac{21}{2}$
$u_3$	0	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$
	0	0	1	2	0	42

And then we must apply the Dual Simplex Method with  $u_3$  as our departing variable and  $u_2$  as our entering variable to get

	$x$	$y$	$u_1$	$u_2$	$u_3$	
$x$	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	5
$y$	0	1	0	0	1	10
$u_3$	0	0	0	1	-2	1
	0	0	1	0	4	40

This tableau represents the (integral!) solution  $x = 5$  and  $y = 10$ , with an objective value of 40. Since the other branch had no feasible solutions, this is an optimal solution.

- 6 Consider the following scenario. Model it as an integer programming problem. Be sure to state explicitly what each of your decision variables  $x_1, x_2, \dots$  represent. Do **not** attempt to solve the problem.

The Rutgers football program is attempting to create their 2006 schedule. They are allowed to play up to four non-conference games, which must be chosen from the list below. Also, they can not play the same opponent twice.

Opponent	Chance of Winning	Revenue (in thousands of \$)
Army	60%	\$400
Buffalo	65%	\$100
Kent State	60%	\$100
New Hampshire	70%	\$80
Notre Dame	20%	\$1,000
Michigan State	25%	\$500
Missouri	20%	\$450
Ohio State	15%	\$850
Navy	50%	\$500
Villanova	55%	\$100

The Scarlet Knights, despite winning the first intercollegiate football game ever played (against Princeton in 1869 by the score of 6-4, only shortly after Princeton beat Rutgers 40-2 in baseball), have played in only one bowl game, the Garden State Bowl in 1978. To give you an idea of how bad they have been doing, Southern Methodist University has been to four bowl games since 1978.

To give Rutgers a shot to break their bowl-drought, they need a schedule for which they can expect at least two victories against their non-conference opponents. (By a principle called Linearity of Expectation, the number of games they can expect to win is the sum of the probabilities of winning each game individually, so for example, if they scheduled Army, Notre Dame, Missouri, and Ohio State, they could expect to win only  $.60 + .20 + .20 + .15 = 1.15$  games.) Beyond this, they would like to maximize revenue.

**Answer:** First we have to decide on variables. There are 10 possible games, so our variables will be  $x_1, x_2, \dots, x_{10}$ , where

$$x_i = \begin{cases} 1 & \text{if Rutgers plays game } i, \\ 0 & \text{otherwise.} \end{cases}$$

(Variables like this are often called “indicator variables.”)

Now we must translate the constraints into mathematical terms. We were told that Rutgers can play at most four non-conference games, so this means that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \leq 4.$$

We were also told that Rutgers must be able to expect at least two victories from their non-conference schedule. By Linearity of Expectation, they can expect

$$.60x_1 + .65x_2 + .60x_3 + .70x_4 + .20x_5 + .25x_6 + .20x_7 + .15x_8 + .50x_9 + .55x_{10}$$

victories, so this must be at least 2.

Finally, we were told to maximize revenue. The revenue from their schedule will be

$$400x_1 + 100x_2 + 100x_3 + 80x_4 + 1000x_5 + 500x_6 + 450x_7 + 850x_8 + 500x_9 + 100x_{10}.$$

Below I have this in the usual table layout

Maximize  $z = 400x_1 + 100x_2 + 100x_3 + 80x_4 + 1000x_5 + 500x_6 + 450x_7 + 850x_8 + 500x_9 + 100x_{10}$   
subject to

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} &\leq 4 \\ .60x_1 + .65x_2 + .60x_3 + .70x_4 + .20x_5 + .25x_6 + .20x_7 + .15x_8 + .50x_9 + .55x_{10} &\geq 2 \\ x_i &\leq 1 \text{ for } i = 1, 2, \dots, 10 \\ x_i &\geq 0 \text{ for } i = 1, 2, \dots, 10\end{aligned}$$

7 Consider the following primal problem

$$\begin{aligned} &\text{Maximize } z = x_1 + 4x_2 + 5x_3 \\ &\text{subject to} \\ &-x_1 + x_2 + x_3 \leq 4 \\ &3x_1 + x_2 + x_3 \leq 16 \\ &\qquad\qquad x_2 \geq 1 \\ &x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

and its dual

$$\begin{aligned} &\text{Minimize } z = 4w_1 + 16w_2 - w_3 \\ &\text{subject to} \\ &-w_1 + 3w_2 \geq 1 \\ &w_1 + w_2 - w_3 \geq 4 \\ &w_1 + w_2 \geq 5 \\ &w_1, \quad w_2, \quad w_3 \geq 0 \end{aligned}$$

Without using the Simplex Method on either problem, find optimal solutions to the two problems from the following list. Explain your reasoning.

$$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} \quad \begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 15 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix} \quad \begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \quad \begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 12 \\ 7 \end{bmatrix}$$

**Answer:** The Theorem we need to use says that if  $\underline{x}$  is a feasible solution to the primal problem and  $\underline{w}$  is a feasible solution to the dual problem and the objective values of both solutions (in their respective problems) are the same, then they are both optimal solutions.

We can start this by ruling out the two solutions with negative entries:

$$\begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 12 \\ 7 \end{bmatrix}$$

For the rest, we can make a chart as follows

possible optimal solution	objective value in primal problem	objective value in dual problem
$\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$	23	infeasible
$\begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$	infeasible	infeasible
$\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 15 \end{bmatrix}$	infeasible	infeasible
$\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$	37	infeasible
$\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$	$14\frac{1}{2}$	37

Note that as soon as we find the two 37s we can stop, since the Weak Duality Theorem (or is it

a corollary to the Weak Duality Theorem?) now tells us that  $\begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$  is an optimal solution to the

primal problem and  $\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$  is an optimal solution to the dual problem.

Another note: I should have been clearer, but you were *not* supposed to use the fact that you knew that each problem had an optimal solution in that list. That is, you weren't supposed to just check feasibility and objective values for each problem independently.