1 Consider the following scenario. Model it as a linear programming problem. Be sure to state explicitly what each of your decision variables $x_1, x_2, \ldots$ represent. Do **not** attempt to solve the problem.

Marion Jones, who won three gold medals in the 2000 olympics, was recently accused by her ex-husband C.J. Hunter of using a mixture of four banned performance-enhancing drugs (often referred to as "stacking") during those olympics: human growth hormone (HGH), anabolic steroids (AS), the endurance booster erythropoietin (EPO), and insulin. Jones had previously been suspended for four years after failing to show up for a random drug test, but her suspension was dropped after an appeal in which she was represented by Johnnie Cochran. Hunter himself tested positive for steroids on four separate occasions in 2000.

Let's say that Jones can use a mixture of three designer drugs, each of which contains a blend of the four substances:

	HGH	AS	EPO	insulin
Type A	10	5	20	5
Type B	3	20	10	0
Type C	20	5	5	5

She would like maximize the total number of units of the four substances she uses, but she doesn't want to get caught or die. So she needs to pass the tests for HGH, AS, and EPO, which require that she ingest at most 60, 40, and 100 units of the three drugs, respectively. Since insulin passes through the body extremely quickly, it is nearly impossible to test for, so Marion doesn't need to worry about that. However, an overdose of insulin causes the body to break down glucose far too quickly, resulting in hypoglycemia, the symptoms of which range from nausea to death. For this reason, Marion should limit her insulin intake to at most 35 units.

Answer: I let my variables be

$$\begin{aligned} a &= \text{amount of Type A} \\ b &= \text{amount of Type B} \\ c &= \text{amount of Type C} \end{aligned}$$

Note that the problem asks to maximize the number of units of HGH+AS+EPO+insulin, so you first have to figure out that Type A has 40 total units of doping agent, Type B has 33, and Type C has 35. This means we are trying to maximize $40a + 33b + 35c$. Very few of you picked up on this. The final answer is then:

$$\begin{array}{l} \text{Maximize } z = 40a + 33b + 35c \\ \text{subject to} \\ \begin{array}{rcrcrcr} 10a & + & 3b & + & 20c & \leq & 60 \\ 5a & + & 20b & + & 5c & \leq & 40 \\ 20a & + & 10b & + & 5c & \leq & 100 \\ 5a & & & + & 5c & \leq & 35 \\ a, & & b, & & c, & \geq 0 & \end{array} \end{array}$$

2 Convert the following linear programming problem into **both** standard form and canonical form. Do **not** attempt to solve the problem.

$$\begin{array}{l}\text{Minimize } z = x_1 - 3x_2 + 7x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} x_1 & - & 2x_2 & + & 4x_3 & \leq & 916 \\ 10x_1 & & & + & 15x_3 & \geq & 1972 \\ x_1 & + & x_2 & + & 6x_3 & = & 123 \\ x_1, & & & & x_3 & \geq & 0 \\ & & x_2 & & & & \text{unconstrained} \end{array}\end{array}$$

Answer: For standard form we get

$$\begin{array}{l}\text{Maximize } z' = -x_1 + 3x_2^+ - 3x_2^- - 7x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcrcl} x_1 & - & 2x_2^+ & + & 2x_2^- & + & 4x_3 & \leq & 916 \\ -10x_1 & & & & & - & 15x_3 & \leq & -1972 \\ x_1 & + & x_2^+ & - & x_2^- & + & 6x_3 & \leq & 123 \\ -x_1 & - & x_2^+ & + & x_2^- & - & 6x_3 & \leq & -123 \\ x_1, & & x_2^+, & & x_2^-, & & x_3 & \geq & 0 \end{array}\end{array}$$

To go to canonical form we could just add slack variables to each of the inequalities above, but it is easier just to add slack variables to the first two inequalities and then replace the last two by the equality they were taking care of: $x_1 + x_2 + 6x_3 = 123$. This gives us:

$$\begin{array}{l}\text{Maximize } z' = -x_1 + 3x_2^+ - 3x_2^- - 7x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcrcrcrcl} x_1 & - & 2x_2^+ & + & 2x_2^- & + & 4x_3 & + & u_1 & & & = & 916 \\ -10x_1 & & & & & - & 15x_3 & & & + & u_2 & = & -1972 \\ x_1 & + & x_2^+ & - & x_2^- & + & 6x_3 & & & & & = & 123 \\ x_1, & & x_2^+, & & x_2^-, & & x_3, & & u_1, & & u_2 & \geq & 0 \end{array}\end{array}$$

3 Solve the following linear programming problem by using the **Extreme Point Theorem**. That is, graph the set of feasible solutions, find the extreme points, and find the optimal solution(s).

$$\begin{array}{l} \text{Maximize } z = 2x + 3y \\ \text{subject to} \\ \begin{array}{rcrcl} x & + & y & \leq & 3 \\ -x & + & y & \leq & 2 \\ x, & & y, & \geq 0 & \end{array} \end{array}$$

Answer: Here are the extreme points and the values of z:

x	y	z
0	2	6
$\frac{1}{2}$	$\frac{5}{2}$	$\frac{17}{2}$
3	0	6

So the optimal solution is $(\frac{1}{2}, \frac{5}{2})$.

4 Solve the following linear programming problem by the **simplex method**. No points will be given for any other method!

$$\begin{array}{l} \text{Maximize } z = 3x_1 + 2x_2 + 4x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ 2x_1 & & & + & 3x_3 & \leq & 5 \\ 2x_1 & + & x_2 & + & 3x_3 & \leq & 7 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

Answer: First, and I can not overstate the importance of this step: **you must convert the problem to canonical form.** So we have to add slack variables u_1, u_2, and u_3 to the above inequalities.
Then the initial tableau is:

	x_1	x_2	x_3	u_1	u_2	u_3	
u_1	1	1	2	1	0	0	4
u_2	2	0	3	0	1	0	5
u_3	2	1	3	0	0	1	7
	-3	-2	-4	0	0	0	0

	x_1	x_2	x_3	u_1	u_2	u_3	
u_1	$-\frac{1}{3}$	1	0	1	$-\frac{2}{3}$	0	$\frac{2}{3}$
x_3	$\frac{2}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{5}{3}$
u_3	0	1	0	0	-1	1	2
	$-\frac{1}{3}$	-2	0	0	$\frac{4}{3}$	0	$\frac{20}{3}$

	x_1	x_2	x_3	u_1	u_2	u_3	
x_2	$-\frac{1}{3}$	1	0	1	$-\frac{2}{3}$	0	$\frac{2}{3}$
x_3	$\frac{2}{3}$	0	1	0	$\frac{1}{3}$	0	$\frac{5}{3}$
u_3	$\frac{1}{3}$	0	0	-1	$-\frac{1}{3}$	1	$\frac{4}{3}$
	-1	0	0	2	0	0	8

	x_1	x_2	x_3	u_1	u_2	u_3	
x_2	0	1	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_1	1	0	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{5}{2}$
u_3	0	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	1	$\frac{1}{2}$
	0	0	$\frac{3}{2}$	2	$\frac{1}{2}$	0	$\frac{21}{2}$

5 Solve the following linear programming problem by the **two-phase simplex method**. No points will be given for any other method!

$$\begin{array}{l} \text{Maximize } z = 3x_1 + x_2 \\ \text{subject to} \\ \begin{array}{rcrcr} x_1 & - & x_2 & \leq & -1 \\ -x_1 & - & x_2 & \leq & -3 \\ 2x_1 & + & x_2 & \leq & 4 \\ x_1, & & x_2, & \geq 0 & \end{array} \end{array}$$

Answer: We need artificial variables in both rows 1 and 2. So we add these artificial variables, and our initial tableau for the first phase of the two-phase simplex method is:
We start with the tableau

	x_1	x_2	u_1	u_2	u_3	y_0	y_1	
y_0	-1	1	-1	0	0	1	0	1
y_1	1	1	0	-1	0	0	1	3
u_3	2	1	0	0	1	0	0	4
	0	0	0	0	0	1	1	0

But this is not our initial tableau, because three are nonzero entries in the objective row under basic variables. After pivotting, we get the following as our initial tableau:

	x_1	x_2	u_1	u_2	u_3	y_0	y_1	
y_0	-1	1	-1	0	0	1	0	1
y_1	1	1	0	-1	0	0	1	3
u_3	2	1	0	0	1	0	0	4
	0	-2	1	1	0	0	0	-4

	x_1	x_2	u_1	u_2	u_3	y_0	y_1	
x_2	-1	1	-1	0	0	1	0	1
y_1	2	0	1	-1	0	-1	1	2
u_3	3	0	1	0	1	-1	0	3
	-2	0	-1	1	0	2	0	-2

The final tableau:

	x_1	x_2	u_1	u_2	u_3	y_0	y_1	
x_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	2
x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	1
u_3	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$-\frac{3}{2}$	0
	0	0	0	0	0	1	1	0

We are now ready to begin the second phase. We start with the tableau

	x_1	x_2	u_1	u_2	u_3	
x_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	1
u_3	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	0
	-3	-1	0	0	0	0

But this is not our initial tableau, because three are nonzero entries in the objective row under basic variables. After pivotting, we get the following as our initial tableau:

	x_1	x_2	u_1	u_2	u_3	
x_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	1
u_3	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	1	0
	0	0	1	-2	0	5

The final tableau:

	x_1	x_2	u_1	u_2	u_3	
x_2	0	1	$-\frac{2}{3}$	0	$\frac{1}{3}$	2
x_1	1	0	$\frac{1}{3}$	0	$\frac{1}{3}$	1
u_2	0	0	$-\frac{1}{3}$	1	$\frac{2}{3}$	0
	0	0	$\frac{1}{3}$	0	$\frac{4}{3}$	5