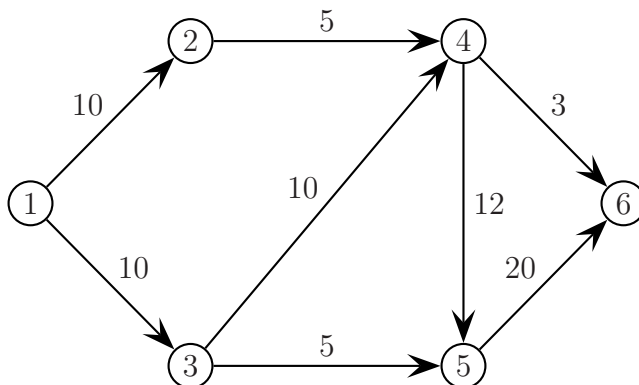
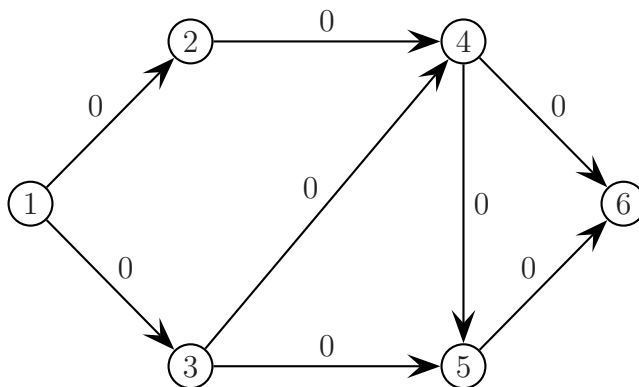


## Homework #7 Solutions

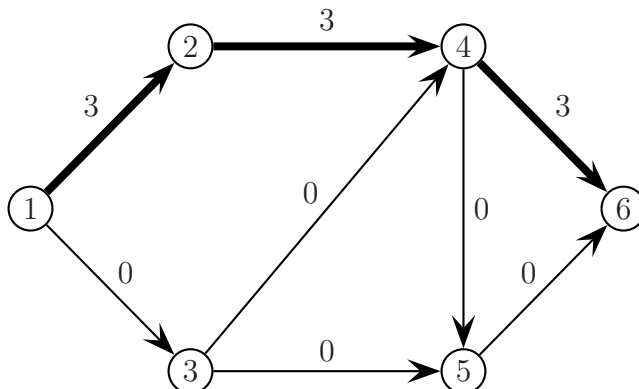
- 1 Find a maximal flow from the source (vertex 1) to the sink (vertex 6) in the following network using the Ford-Fulkerson Algorithm. Prove that your flow is optimal by exhibiting a cut with the same capacity.



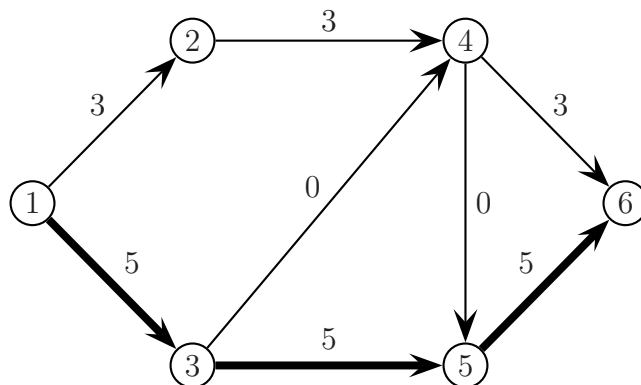
**Answer:** Let's start with the flow. At the beginning we are piping nothing through:



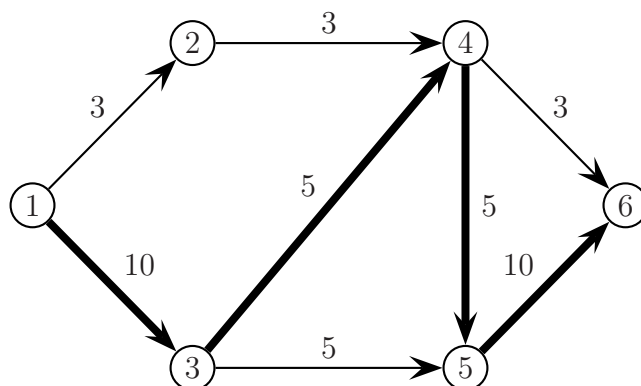
Let's add the augmenting path  $1 \rightarrow 2 \rightarrow 4 \rightarrow 6$ , which we can pipe 3 down:



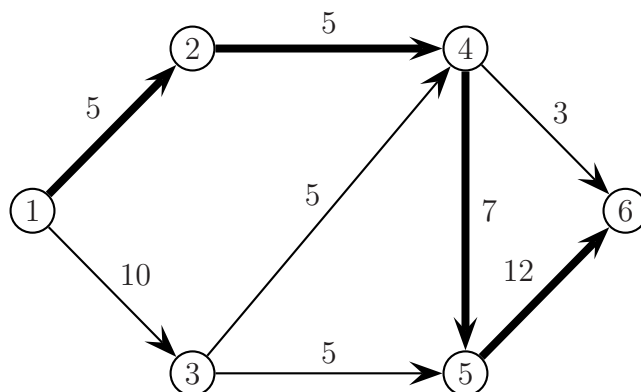
Now we can add the augmenting path  $1 \rightarrow 3 \rightarrow 5 \rightarrow 6$ , which we can send 5 units down.



We can also send 5 units down the path  $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6$  to get the flow

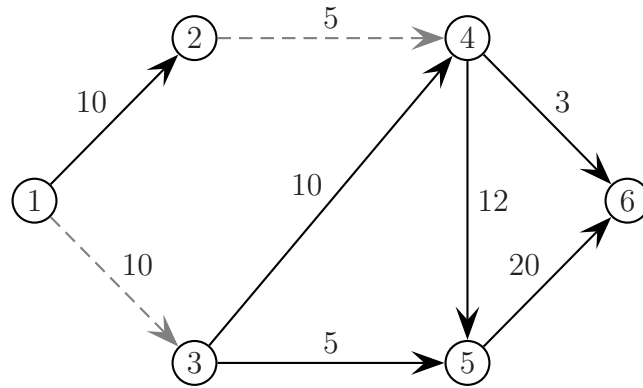


Then we can send 2 units down the path  $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$  to get

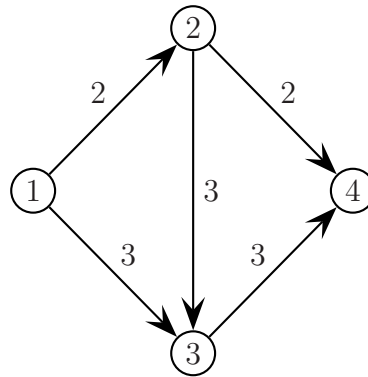


This flow gets 15 units of material from 1 to 6. It is indeed a maximal flow, because there is

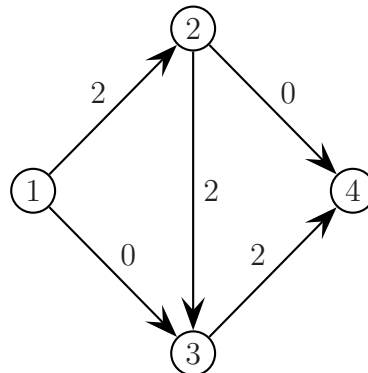
a cut of capacity 15:



2 Consider the following network with source 1 and sink 4.

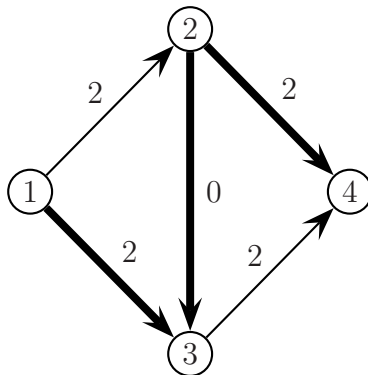


Extend the following flow to an optimal flow using the Ford-Fulkerson Algorithm. Then prove that you have achieved an optimal flow by exhibiting a cut with the same capacity.

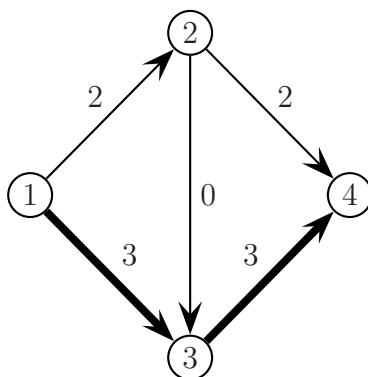


**Answer:** Let's add the augmenting path  $1 \rightarrow 3 \rightarrow 2 \rightarrow 4$ . We can send 3 units from 1 to 3, 2 units from 3 to 2 (since we are traveling that arc in the "wrong" direction, we can only send as many units as are flowing in the "right" direction), and 2 units from 2 to 4. So we

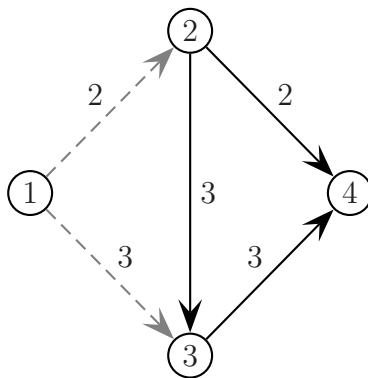
can send 2 units down this path, which gives us



We can now add the augmenting path  $1 \rightarrow 3 \rightarrow 4$ , which we can ship 1 unit down.



This gives a flow of 5 units, which is maximal because of the cut



3 Use the algorithm from Section 5.5 of your book (which is also the algorithm I presented in



We once again consider our options and find that the shortest option is to go to node 3 in 9 steps through node 4, which gives the following chart:

1	2 - 3	3 - 9	4 - 7	5	6	7 - 3	8	9
<u>2 - 3</u>	3 - 8	5 - 3	<u>3 - 2</u>	6 - 6	9 - 6	4 - 5	7 - 1	
4 - 9	<u>4 - 4</u>		5 - 6	8 - 4			9 - 5	
<u>7 - 3</u>								

Our shortest option now is node 5 in 12 steps:

1	2 - 3	3 - 9	4 - 7	5 - 12	6	7 - 3	8	9
<u>2 - 3</u>	3 - 8	<u>5 - 3</u>	<u>3 - 2</u>	6 - 6	9 - 6	4 - 5	7 - 1	
4 - 9	<u>4 - 4</u>		5 - 6	8 - 4			9 - 5	
<u>7 - 3</u>								

Then node 8 in 16 steps:

1	2 - 3	3 - 9	4 - 7	5 - 12	6	7 - 3	8 - 16	9
<u>2 - 3</u>	3 - 8	<u>5 - 3</u>	<u>3 - 2</u>	6 - 6	9 - 6	4 - 5	7 - 1	
4 - 9	<u>4 - 4</u>		5 - 6	<u>8 - 4</u>			9 - 5	
<u>7 - 3</u>								

Then node 6 in 18 steps:

1	2 - 3	3 - 9	4 - 7	5 - 12	6 - 18	7 - 3	8 - 16	9
<u>2 - 3</u>	3 - 8	<u>5 - 3</u>	<u>3 - 2</u>	<u>6 - 6</u>	9 - 6	4 - 5	7 - 1	
4 - 9	<u>4 - 4</u>		5 - 6	<u>8 - 4</u>			9 - 5	
<u>7 - 3</u>								

And finally, we go to node 9 in 21 steps:

1	2 - 3	3 - 9	4 - 7	5 - 12	6 - 18	7 - 3	8 - 16	9 - 21
<u>2 - 3</u>	3 - 8	<u>5 - 3</u>	<u>3 - 2</u>	<u>6 - 6</u>	9 - 6	4 - 5	7 - 1	
4 - 9	<u>4 - 4</u>		5 - 6	<u>8 - 4</u>			<u>9 - 5</u>	
<u>7 - 3</u>								

We can read this chart backwards to find that our shortest route was

$$1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 8 \rightarrow 9.$$

- 4 Use the Hungarian Method to solve the following integer programming problem. Be careful to note that this is a maximization problem.

The Rutgers football program needs to assign positions to four new players. Each can play quarterback (QB), tight-end (TE), wide receiver (WR), and cornerback (CB) moderately well, but the team can have only one new player at each of these positions. The chart below shows the value that each player will add to each position.

	QB	TE	WR	CB
Ben	5	8	4	4
Bob	3	3	4	3
Jim	4	5	3	5
Ross	3	3	5	6

How should the team assign positions to these four players in order to maximize the total value added?

**Answer:** First we need to convert this problem to a minimization problem. Doing so multiplies our cost matrix by  $-1$  to give

$$\begin{bmatrix} -5 & -8 & -4 & -4 \\ -3 & -3 & -4 & -3 \\ -4 & -5 & -3 & -5 \\ -3 & -3 & -5 & -6 \end{bmatrix}$$

Since this matrix has negative entries, we add the absolute value of the most negative entry to everything in the matrix to get

$$\begin{bmatrix} -5 & -8 & -4 & -4 \\ -3 & -3 & -4 & -3 \\ -4 & -5 & -3 & -5 \\ -3 & -3 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 4 & 4 \\ 5 & 5 & 4 & 5 \\ 4 & 3 & 5 & 3 \\ 5 & 5 & 3 & 2 \end{bmatrix}$$

Now we subtract the smallest entry in each column from everything in that column to get

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 5 & 1 & 3 \\ 1 & 3 & 2 & 1 \\ 2 & 5 & 0 & 0 \end{bmatrix}$$

And we do the same to the rows:

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 4 & 0 & 2 \\ 0 & 2 & 1 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix}$$

This gives us the following assignment:

$$\begin{bmatrix} 0 & \textcircled{0} & 1 & 2 \\ 1 & 4 & \textcircled{0} & 2 \\ \textcircled{0} & 2 & 1 & 0 \\ 2 & 5 & 0 & \textcircled{0} \end{bmatrix}$$