Math 354 Summer 2004

Homework #5 Solutions

Please write your answers on a separate sheet of paper, and include at least some intermediate steps and English words.

This homework covers sections 3.1-3.4.

1 Consider the linear programming problem

$$\begin{array}{l} \text{Maximize } z = 5x_1 + 10x_2 \\ \text{subject to} \\ \begin{array}{rcrcl} x_1 & + & 3x_2 & \le & 50 \\ 4x_1 & + & 2x_2 & \le & 60 \\ x_1 & & & \le & 5 \\ x_1, & & x_2 & \ge & 0 \end{array} \end{array}$$

(a) State the dual of the preceding LPP.

(b) Given that $(5, 15)$ is an optimal solution to this LPP, use the Duality Theorem and the principle of complementary slackness to find optimal solution to the dual.

Answer: The dual of this LPP is:

$$\begin{array}{l} \text{Minimize } z' = 50w_1 + 60w_2 + 5w_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} w_1 & + & 4w_2 & + & w_3 & \ge & 5 \\ 3w_1 & + & 2w_2 & & & \ge & 10 \\ w_1, & & w_2, & & w_3 & \ge & 0 \end{array} \end{array}$$

For part (b), let's begin by recalling the principle:

Principle of Complementary Slackness: Let $\underline{x}$ be an optimal solution to an LPP and let $\underline{w}$ be an optimal solution to the dual problem. Suppose that when we plug $\underline{x}$ into the ith inequality of the primal problem has slack (i.e., is not tight). Then the ith component of $\underline{w}$ is 0.

We can also apply the dual of this principle, which states that if we plug $\underline{w}$ into the dual problem and see that the jth inequality of the this problem has slack, then the jth component of $\underline{x}$ is 0.

Plugging $(5, 15)$ into the primal problem, we see that the second inequality becomes $4(5) + 2(15) \le 60$, which has slack. Therefore $w_2 = 0$ by the Principle of Complementary Slackness.

Then we look at the second inequality in the dual problem. It states that $3w_1 + 2w_2 \ge 10$. But now we know that $w_2 = 0$, so this says $3w_1 \ge 10$. If this inequality had slack, then the dual version of the Principle of Complementary Slackness would say that $x_2 = 0$. But this can't be the case, because $x_2 = 15 \ne 0$. Therefore this inequality is tight, so $3w_1 = 10$, and thus $w_1 = 10/3$.

Similarly, the first inequality in the dual problem can't have slack, so substituting $w_1 = 10/3$ and $w_2 = 0$, we see that

$$\frac{10}{3} + w_3 = 5,$$

so $w_3 = 5/3$.

Therefore $w_1 = 10/3$, $w_2 = 0$, and $w_3 = 5/3$ gives an optimal solution to the dual problem.

2 The tableau below represents a solution to a linear programming problem that satisfies the optimality criterion, but is infeasible. Use the dual simplex method to restore feasibility.

		5	6	0	0	0	0	
$\underline{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	$\underline{x}_B$
5	x_1	1	0	0	-1	1	0	4
6	x_2	0	1	0	1	$-\frac{2}{3}$	0	$\frac{10}{3}$
0	x_3	0	0	1	7	-8	0	2
0	x_6	0	0	0	0	$-\frac{1}{3}$	1	$-\frac{1}{3}$
		0	0	0	1	1	0	40

Answer: We choose x_6 as our departing variable, and then the only choice we have for an entering variable is x_5. After one pivot, the final tableau is

		5	6	0	0	0	0	
$\underline{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	$\underline{x}_B$
5	x_1	1	0	0	-1	0	3	3
6	x_2	0	1	0	1	0	-2	4
0	x_3	0	0	1	7	0	-24	10
0	x_6	0	0	0	0	1	-3	1
		0	0	0	1	0	3	39

This corresponds to the basic feasible solution $\underline{x} = [3\ 4\ 10\ 0\ 1\ 0]^T$.

3 In class on Monday we divided linear programming problems into three classes:

(A) No feasible solutions

(B) Feasible solutions, finite optimal objective value

(C) Feasible solutions, arbitrarily large objective values

The two-phase simplex method can solve all the problems of type (B).

What would happen if I made a typo on the final exam and asked you to use the two-phase method on a problem of type (A), that is, a problem with no feasible solutions? How far could you get?

Answer: You would be able to set up the first phase of the two-phase method and solve it. However, at least one of the artificial variables would remain positive, so you wouldn't have a basic feasible solution to the original problem with which to begin phase 2.