Math 354 Summer 2004

Homework #4 Solutions

1 Find the dual of the following linear programming problem.

$$\begin{array}{l} \text{Maximize } z = 6x_1 + 6x_2 + 8x_3 + 9x_4 \\ \text{subject to} \\ \begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & x_4 & \geq & 3 \\ 2x_1 & + & x_2 & + & 4x_3 & + & 9x_4 & \geq & 8 \\ x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array} \end{array}$$

Answer: First we have to convert this problem to standard form. That gives

$$\begin{array}{l} \text{Maximize } z = 6x_1 + 6x_2 + 8x_3 + 9x_4 \\ \text{subject to} \\ \begin{array}{rcrcrcrcr} -x_1 & - & 2x_2 & - & x_3 & - & x_4 & \leq & -3 \\ -2x_1 & - & x_2 & - & 4x_3 & - & 9x_4 & \leq & -8 \\ x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array} \end{array}$$

Then we can find the dual as usual:

$$\begin{array}{l} \text{Minimize } z' = -3w_1 - 8w_2 \\ \text{subject to} \\ \begin{array}{rcrcr} -w_1 & - & 2w_2 & \geq & 6 \\ -2w_2 & - & w_2 & \geq & 6 \\ -w_1 & - & 4w_2 & \geq & 8 \\ -w_1 & - & 9w_2 & \geq & 9 \\ w_1, & & w_2 & \geq & 0 \end{array} \end{array}$$

To be honest, this isn't exactly the problem I meant to ask, because the primal problem rather obviously has feasible solutions with arbitrarily large objective values. Note that the dual problem has no feasible solutions, as guaranteed by a theorem in the text.

2 Suppose that a linear programming problem has no feasible solutions. What does the Duality Theorem say about solutions to the dual problem?

Answer: Part (a) of the Duality Theorem on page 174 of your text says: "If either the primal or dual problem has a feasible solution with a finite optimal objective value, the the other problem has a feasible solution with the same objective value."

I claim that this implies, under the hypotheses of this problem, that the dual problem does not have a feasible solution with a finite optimal objective value. To see this, suppose to the contrary that the dual problem does have a feasible solution with... Then by the Duality Theorem, the primal problem has a feasible solution with the same objective value, but this is a contradiction, because we assumed that the primal problem doesn't have any feasible solutions. This contradiction proves the claim.

Note that this leaves two possibilities for the dual problem: it can either have no feasible solutions, or it can have feasible solutions with arbitrarily large objective function values.

(For this last case, several of you said that the "dual problem is unbounded." This seems quite ambiguous - do you mean the set of feasible solutions to the problem is unbounded? (which would be wrong) - or that the objective values are unbounded? (which would be correct) - or, were you just copying out of that table in your book?) In either case, the dual problem does not have a feasible solution with an *optimal* objective value.