Math 354 Summer 2004

Homework #3 Solutions

Please write your answers on a separate sheet of paper, and include at least some intermediate steps and English words.

This homework is intended to cover sections 2.1 through 2.3.

1 Solve the following linear programming problem using the simplex method.

$$\begin{array}{l} \text{Maximize } z = 2x + 5y \\ \text{subject to} \\ \begin{array}{rcrcr} 3x & + & 5y & \leq & 8, \\ 2x & + & 7y & \leq & 12, \end{array} \\ x \geq 0,\ y \geq 0 \end{array}$$

Answer: First convert the problem to canonical form.

$$\begin{array}{l} \text{Maximize } z = 2x + 5y \\ \text{subject to} \\ \begin{array}{rcrcrcrcr} 3x & + & 5y & + & u & & & = & 8, \\ 2x & + & 7y & & & + & v & = & 12, \end{array} \\ x \geq 0,\ y \geq 0,\ u \geq 0,\ v \geq 0 \end{array}$$

The initial tableau is

	x	y	u	v	
u	3	5	1	0	8
v	2	7	0	1	12
	-2	-5	0	0	0

After one iteration (u departs, y enters) we get the final tableau:

	x	y	u	v	
y	$\frac{3}{5}$	1	$\frac{1}{5}$	0	$\frac{8}{5}$
v	$-\frac{11}{5}$	0	$-\frac{7}{5}$	1	$\frac{4}{5}$
	1	0	1	0	8

This gives the final answer: $x = 0$, $y = 8/5$, $u = 0$, $v = 4/5$, with an optimal objective function value of 8.

2 Consider the following tableau, which arose in solving a linear programming problem by the simplex method.

x_1	x_2	x_3	u	v	w	
1	5	2	0	0	3	20
0	2	4	1	0	-4	6
0	2	-1	0	1	3	12
0	-5	-3	0	0	3	12

(a) Identify the basic feasible solution and basic variables in this tableau.

(b) Compute the next tableau using the simplex method.

(c) Identify the basic feasible solution and basic variables in the tableau in (b).

Answer: Part (a) asks us to replace the left-hand column. The basic variables are x_1, u, and v, in that order. So this tableau corresponds to the basic feasible solution $x_1 = 20$, $x_2 = 0$, $x_3 = 0$, $u = 6$, $v = 12$, and $w = 0$, with an objective function value of 12.

For part (b) we perform one iteration; x_2 enters and u departs to give us

	x_1	x_2	x_3	u	v	w	
x_1	1	0	-8	$-\frac{5}{2}$	0	13	5
x_2	0	1	2	$\frac{1}{2}$	0	-2	3
v	0	0	-5	-1	1	7	6
	0	0	7	$\frac{5}{2}$	0	-7	27

Part (c) just asks us to read off the solution this corresponds to. It is $x_1 = 5$, $x_2 = 3$, $x_3 = 0$, $u = 0$, $v = 6$, and $w = 0$, with an objective function value of 27.

If you are curious, one more iteration of the simplex method would have found the optimal solution: $x_1 = 0$, $x_2 = 49/13$, $x_3 = 0$, $u = 0$, $v = 43/13$, and $w = 5/13$, with an optimal objective function value of $386/13$.

3 Solve the following linear programming problem using the two-phase method.

$$
\begin{array}{l}
\text{Maximize } z = 3x_1 - x_2 + 2x_3 + 4x_4 \\
\text{subject to} \\
\begin{array}{rcrcrcrcr}
 & & x_2 & + & 7x_3 & + & 2x_4 & \geq & 3, \\
x_1 & + & 2x_2 & + & x_3 & & & = & 9, \\
2x_1 & + & 3x_2 & + & x_3 & - & 4x_4 & \leq & 7,
\end{array} \\
x_j \geq 0,\ \ j = 1, 2, 3, 4.
\end{array}
$$

Answer: Let me apologize once again for this assigning this seemingly endless problem... but, here's the solution.

We need artificial variables in both rows 1 and 2. So we add these artificial variables, and we are ready to begin the first phase of the two-phase simplex method. We start with the tableau

	x_1	x_2	x_3	x_4	u_1	u_2	y_0	y_1	
y_0	0	1	7	2	-1	0	1	0	3
y_1	1	2	1	0	0	0	0	1	9
u_2	2	3	1	-4	0	1	0	0	7
	0	0	0	0	0	0	1	1	0

But this is not our initial tableau, because three are nonzero entries in the objective row under basic variables. After pivotting, we get the following as our initial tableau:

	x_1	x_2	x_3	x_4	u_1	u_2	y_0	y_1	
y_0	0	1	7	2	-1	0	1	0	3
y_1	1	2	1	0	0	0	0	1	9
u_2	2	3	1	-4	0	1	0	0	7
	-1	-3	-8	-2	1	0	0	0	-12

	x_1	x_2	x_3	x_4	u_1	u_2	y_0	y_1	
x_3	0	$\frac{1}{7}$	1	$\frac{2}{7}$	$-\frac{1}{7}$	0	$\frac{1}{7}$	0	$\frac{3}{7}$
y_1	1	$\frac{13}{7}$	0	$-\frac{2}{7}$	$\frac{1}{7}$	0	$-\frac{1}{7}$	1	$\frac{60}{7}$
u_2	2	$\frac{20}{7}$	0	$-\frac{30}{7}$	$\frac{1}{7}$	1	$-\frac{1}{7}$	0	$\frac{46}{7}$
	-1	$-\frac{13}{7}$	0	$\frac{2}{7}$	$-\frac{1}{7}$	0	$\frac{8}{7}$	0	$-\frac{60}{7}$

	x_1	x_2	x_3	x_4	u_1	u_2	y_0	y_1	
x_3	$-\frac{1}{10}$	0	1	$\frac{1}{2}$	$-\frac{3}{20}$	$-\frac{1}{20}$	$\frac{3}{20}$	0	$\frac{1}{10}$
y_1	$-\frac{3}{10}$	0	0	$\frac{5}{2}$	$\frac{1}{20}$	$-\frac{13}{20}$	$-\frac{1}{20}$	1	$\frac{43}{10}$
x_2	$\frac{7}{10}$	1	0	$-\frac{3}{2}$	$\frac{1}{20}$	$\frac{7}{20}$	$-\frac{1}{20}$	0	$\frac{23}{10}$
	$\frac{3}{10}$	0	0	$-\frac{5}{2}$	$-\frac{1}{20}$	$\frac{13}{20}$	$\frac{21}{20}$	0	$-\frac{43}{10}$

	x_1	x_2	x_3	x_4	u_1	u_2	y_0	y_1	
x_4	$-\frac{1}{5}$	0	2	1	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{3}{10}$	0	$\frac{1}{5}$
y_1	$\frac{1}{5}$	0	-5	0	$\frac{4}{5}$	$-\frac{2}{5}$	$-\frac{4}{5}$	1	$\frac{19}{5}$
x_2	$\frac{2}{5}$	1	3	0	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{13}{5}$
	$-\frac{1}{5}$	0	5	0	$-\frac{4}{5}$	$\frac{2}{5}$	$\frac{9}{5}$	0	$-\frac{19}{5}$

The final tableau:

	x_1	x_2	x_3	x_4	u_1	u_2	y_0	y_1	
x_4	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	0	$-\frac{1}{4}$	0	$\frac{3}{8}$	$\frac{13}{8}$
u_1	$\frac{1}{4}$	0	$-\frac{25}{4}$	0	1	$-\frac{1}{2}$	-1	$\frac{5}{4}$	$\frac{19}{4}$
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	$\frac{9}{2}$
	0	0	0	0	0	0	1	1	0

We are now ready to begin the second phase. We start with the tableau

	x_1	x_2	x_3	x_4	u_1	u_2	
x_4	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	0	$-\frac{1}{4}$	$\frac{13}{8}$
u_1	$\frac{1}{4}$	0	$-\frac{25}{4}$	0	1	$-\frac{1}{2}$	$\frac{19}{4}$
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	$\frac{9}{2}$
	-3	1	-2	-4	0	0	0

But this is not our initial tableau, because three are nonzero entries in the objective row under basic variables. After pivotting, we get the following as our initial tableau:

	x_1	x_2	x_3	x_4	u_1	u_2	
x_4	$-\frac{1}{8}$	0	$\frac{1}{8}$	1	0	$-\frac{1}{4}$	$\frac{13}{8}$
u_1	$\frac{1}{4}$	0	$-\frac{25}{4}$	0	1	$-\frac{1}{2}$	$\frac{19}{4}$
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	0	$\frac{9}{2}$
	-4	0	-2	0	0	-1	2

The next tableau shows that no finite optimal solution to our problem exists. Kind of anti-climactic, huh?

	x_1	x_2	x_3	x_4	u_1	u_2	
x_4	0	$\frac{1}{4}$	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	$\frac{11}{4}$
u_1	0	$-\frac{1}{2}$	$-\frac{13}{2}$	0	1	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	2	1	0	0	0	9
	0	8	2	0	0	-1	38