Math 354 Summer 2004

Homework #2 Solutions

1 Sketch the set of feasible solutions to the following set of inequalities

$$\begin{aligned} -x+y &\leq 2, \\ 2x+y &\leq 2, \\ x &\geq 0, \\ y &\leq 1. \end{aligned}$$

Answer: You all got it right, and besides, graphs are hard to draw.

2 Prove that a hyperplane (defined on page 72, a hyperplane is a set of the form $\{\underline{x} : \underline{a}^T \underline{x} = b\}$ for some vector $\underline{a}$ and real number b) is a convex set (defined on page 79).

Proof: Let H be a hyperplane. We know that H is of the form $\{\underline{x} : \underline{a}^T \underline{x} = b\}$ for some $\underline{a}$ and b. We want to show that H is convex, i.e., for any two points $\underline{x}_1$ and $\underline{x}_2$ in H and any point $\underline{x}$ on the line segment between $\underline{x}_1$ and $\underline{x}_2$, we want to show that $\underline{x}$ also lies in H. Now take two points $\underline{x}_1$ and $\underline{x}_2$ in H and let $\underline{x}$ lie on the line segment between $\underline{x}_1$ and $\underline{x}_2$. We know that $\underline{x} = \lambda \underline{x}_1 + (1-\lambda)\underline{x}_2$ for some λ between 0 and 1. Now we just check the hyperplane condition for $\underline{x}$:

$$\begin{aligned} \underline{a}^T \underline{x} &= \underline{a}^T \left(\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2\right), \\ &= \lambda \underline{a}^T \underline{x}_1 + (1-\lambda)\underline{a}^T \underline{x}_2, \\ &= \lambda b + (1-\lambda) b, \\ &= b, \end{aligned}$$

and this shows that $\underline{x}$ does indeed lie on the hyperplane, proving the claim.

3 Prove that the intersection of two convex sets is a convex set.

Proof: Let A and B be convex sets. We want to show that $A \cap B$ is also convex. Take $\underline{x}_1, \underline{x}_2 \in A \cap B$, and let $\underline{x}$ lie on the line segment between these two points. Then $\underline{x} \in A$ because A is convex, and similarly, $\underline{x} \in B$ because B is convex. Therefore $\underline{x} \in A \cap B$, as desired.

4 Prove that the intersection of any finite number of convex sets is a convex set. Hint: this should follow from the previous problem; for example, to show that $A_1 \cap A_2 \cap A_3$ is convex, write it as $(A_1 \cap A_2) \cap A_3$.

Proof: Let $A_1, A_2, \ldots, A_n$ be convex sets. Then

$$A_1 \cap A_2 \cap \ldots \cap A_n = (\ldots((A_1 \cap A_2) \cap A_3) \ldots) \cap A_n).$$

By the previous problem, $A_1 \cap A_2$ is convex, so $((A_1 \cap A_2) \cap A_3)$ is also convex, etc.

To write up a more formal proof, use induction.

5 Let A be an $m \times n$ matrix and let $\underline{b}$ be a vector in $\mathbb{R}^m$. Prove that the set of vectors $\underline{x}$ in $\mathbb{R}^n$ satisfying $A\underline{x} = \underline{b}$ is a convex set. Hints: note that the empty set, $\emptyset$, is trivially a convex set (at least according to me; whether or not our book agrees on this is unclear), and try to use the previous three problems.

Proof: First, sorry about the misleading hint. You can certainly do it that way, but the easier way is just to copy the proof from 2:

Choose $\underline{x}_1$ and $\underline{x}_2$ that satisfy $A\underline{x}_1 = \underline{b}$ and $A\underline{x}_2 = \underline{b}$. Let $\underline{x}$ lie on the line segment between $\underline{x}_1$ and $\underline{x}_2$, we want to show that $\underline{x}$ also satisfies $A\underline{x} = \underline{b}$. We know that $\underline{x} = \lambda\underline{x}_1 + (1-\lambda)\underline{x}_2$ for some λ between 0 and 1. Now we have

$$\begin{aligned} A\underline{x} &= A\left(\lambda\underline{x}_1 + (1-\lambda)\underline{x}_2\right), \\ &= \lambda A\underline{x}_1 + (1-\lambda)A\underline{x}_2, \\ &= \lambda\underline{b} + (1-\lambda)\underline{b}, \\ &= \underline{b}, \end{aligned}$$

and this shows that $\underline{x}$ also lies in this set, proving the claim.

6 Find the extreme points of the set of feasible solutions for the following optimization problem and then find the optimal solution(s).

$$\begin{array}{l} \text{Maximize } z = 2x + 3y \\ \text{subject to} \\ \begin{array}{rcrcl} 3x & + & y & \leq & 6, \\ x & + & y & \leq & 4, \\ x & + & 2y & \leq & 6, \\ x & & & \geq & 0, \\ & & y & \geq & 0. \end{array} \end{array}$$

Answer: Once again, graphs are a pain, so I'll skip that part. Here's a table:

extreme points	$z = 2x + 3y$
$(0, 0)$	0
$(0, 3)$	9
$(2, 0)$	4
$(\frac{6}{5}, \frac{12}{5})$	$\frac{48}{5} = 9.6$

So, the optimal solution is $(6/5, 12/5)$.