Math 354 Summer 2004 Due Monday 7/19/2004

Homework #2

Please write your answers on a separate sheet of paper, and include at least some intermediate steps and English words.

This homework is intended to cover sections 1.3 and 1.4.

1 Sketch the set of feasible solutions to the following set of inequalities

$$\begin{aligned} -x+y &\leq 2, \\ 2x+y &\leq 2, \\ x &\geq 0, \\ y &\leq 1. \end{aligned}$$

2 Prove that a hyperplane (defined on page 72, a hyperplane is a set of the form $\{\underline{x} : \underline{a}^T\underline{x} = b\}$ for some vector $\underline{a}$ and real number b) is a convex set (defined on page 79).

How you might start your answer to Problem 2, but you certainly don't have to start this way:

Let H be a hyperplane. We know that H is of the form $\{\underline{x} : \underline{a}^T\underline{x} = b\}$ for some $\underline{a}$ and b. We want to show that H is convex, i.e., for any two points $\underline{x}_1$ and $\underline{x}_2$ in H and any point $\underline{x}$ on the line segment between $\underline{x}_1$ and $\underline{x}_2$, we want to show that $\underline{x}$ also lies in H. Now take two points $\underline{x}_1$ and $\underline{x}_2$ in H and let $\underline{x}$ lie on the line segment between $\underline{x}_1$ and $\underline{x}_2$. We know that $\underline{x} = \lambda\underline{x}_1 + (1-\lambda)\underline{x}_2$ for some λ between 0 and 1... (now you just have to show that $\underline{x}$ lies in H and conclude)

3 Prove that the intersection of two convex sets is a convex set. Hint: this shouldn't be too hard.

4 Prove that the intersection of any finite number of convex sets is a convex set. Hint: this should follow from the previous problem; for example, to show that $A_1 \cap A_2 \cap A_3$ is convex, write it as $(A_1 \cap A_2) \cap A_3$.

5 Let A be an $m \times n$ matrix and let $\underline{b}$ be a vector in $\mathbb{R}^m$. Prove that the set of vectors $\underline{x}$ in $\mathbb{R}^n$ satisfying $A\underline{x} = \underline{b}$ is a convex set. Hints: note that the empty set, $\emptyset$, is trivially a convex set (at least according to me; whether or not our book agrees on this is unclear), and try to use the previous three problems.

6 Find the extreme points of the set of feasible solutions for the following optimization problem and then find the optimal solution(s).

Maximize $z = 2x + 3y$
subject to

$$\begin{aligned} 3x + y &\leq 6, \\ x + y &\leq 4, \\ x + 2y &\leq 6, \\ x &\geq 0, \\ y &\geq 0. \end{aligned}$$