Math 354 Summer 2004

Homework #1 Solutions

The administrator of a $200,000 trust fund set up by Mr. Smith's will must adhere to certain guidelines. The total amount of $200,000 need not be fully invested at any one time. The money may be invested in three different types of securities: a utilities stock paying a 9% dividend, an electronics stock paying a 4% dividend, and a bond paying 5% interest. Suppose that the amount invested in the stocks cannot be more than half the total amount invested; the amount invested in the utilities stock cannot exceed $40,000; and the amount invested in the bond must be at least $70,000. What investment policy should be pursued to maximize the return?

1 Set up a linear programming model of this situation.

Answer: First we have to give all the quantities in the problem variable names. I'm going to choose

$$\begin{aligned} a &= \text{amount in utilities stock} \\ b &= \text{amount in electronics stock} \\ c &= \text{amount in bond} \end{aligned}$$

Now we have to translate all of the conditions in the problem into mathematical statements. So

"The total amount of $200,000 need not be fully invested at any one time."

becomes

$$a + b + c \leq 200,000,$$

and

"the amount invested in the stocks cannot be more than half the total amount invested"

becomes

$$\begin{aligned} a + b &\leq \frac{1}{2}\left(\text{total amount invested}\right), \\ &= \frac{1}{2}\left(a + b + c\right). \end{aligned}$$

Note that this is **not** $a + b \leq 100,000$, because the total amount invested is $a + b + c$, and is not necessarily $200,000$. We should also put all the variables in this inequality on the left, getting

$$\frac{a + b - c}{2} \leq 0.$$

Continuing in this manner, we convert

"the amount invested in the utilities stock cannot exceed $40,000"

into

$$a \leq 40,000,$$

and

"the amount invested in the bond must be at least $70,000" into

$$c \geq 70,000.$$

We also have the constraints that $a \geq 0$, $b \geq 0$, and $c \geq 0$, because we can not invest a negative amount of money in any of the offerings. (Never mind the fact that in real life you could invest a negative amount of money in a stock.)

Putting this all together, our linear optimization problem is:

$$\begin{array}{l} \text{Maximize } z = 1.09a + 1.04b + 1.05c \\ \text{subject to} \\ \begin{array}{ccccccc} a & + & b & + & c & \leq & 200,000, \\ \frac{1}{2}a & + & \frac{1}{2}b & - & \frac{1}{2}c & \leq & 0, \\ a & & & & & \leq & 40,000, \\ & & & & c & \geq & 70,000, \end{array} \\ a \geq 0, b \geq 0, c \geq 0. \end{array}$$

2 Convert this model to standard form.

Answer: There is only one constraint in our answer to #1 above that must be changed to put it into standard form, and that is $c \geq 70,000$. We multiply both sides of this inequality by -1 to switch the inequality and get

$$-c \leq -70,000.$$

This gives the final answer of

$$\begin{array}{l} \text{Maximize } z = 1.09a + 1.04b + 1.05c \\ \text{subject to} \\ \begin{array}{ccccccc} a & + & b & + & c & \leq & 200,000, \\ \frac{1}{2}a & + & \frac{1}{2}b & - & \frac{1}{2}c & \leq & 0, \\ a & & & & & \leq & 40,000, \\ & & & & -c & \leq & -70,000, \end{array} \\ a \geq 0, b \geq 0, c \geq 0. \end{array}$$

(Good review questions: first, what would you have done with a constraint that said something like "$a + b = 100,000$," and secondly, what would you have done if we didn't have $a \geq 0$?)

3 Convert this model to canonical form.

Answer: Here we have to change all the $\leq$s to $=$s, but adding slack variables. For example, $a + b + c \leq 200,000$ becomes $a + b + c + u_1 = 200,000$, where $u_1 \geq 0$. Once you do this to all the constraints, you get

$$\begin{array}{l} \text{Maximize } z = 1.09a + 1.04b + 1.05c \\ \text{subject to} \\ \begin{array}{ccccccccccccccc} a & + & b & + & c & + & u_1 & & & & & & & = & 200,000, \\ \frac{1}{2}a & + & \frac{1}{2}b & - & \frac{1}{2}c & & & + & u_2 & & & & & = & 0, \\ a & & & & & & & & & + & u_3 & & & = & 40,000, \\ & & & & -c & & & & & & & + & u_4 & = & -70,000, \end{array} \\ a \geq 0, b \geq 0, c \geq 0, u_1 \geq 0, u_2 \geq 0, u_3 \geq 0, u_4 \geq 0. \end{array}$$