Math 354 Summer 2004

Review problems for the Final Exam

1 Consider (again) the following linear programming problem from Midterm #2.

$$\begin{array}{l} \text{Maximize } z = 4x_1 + 6x_2 + 2x_3 \\ \text{subject to} \\ \begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & \leq & 10 \\ x_1 & + & 4x_2 & & & \leq & 15 \\ x_1 & & & + & x_3 & \leq & 6 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

Applying the Simplex Method to this problem yields the following final tableau

		4	6	2	0	0	0	
$\underline{c}_B$		x_1	x_2	x_3	u_1	u_2	u_3	
0	u_1	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
6	x_2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
4	x_1	1	0	1	0	0	1	6
		0	0	$\frac{1}{2}$	0	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{75}{2}$

This tableau represents the optimal solution

$$\underline{x} = \begin{bmatrix} 6 \\ \frac{9}{4} \\ 0 \end{bmatrix}$$

Suppose now that the problem is changed to "Maximize $z = (4 + \Delta)x_1 + 6x_2 + 2x_3$" for some real number Δ. For what range of Δ is

$$\underline{x} = \begin{bmatrix} 6 \\ \frac{9}{4} \\ 0 \end{bmatrix}$$

still an optimal solution for the new problem? (Hint: you may want to try a specific value or two for Δ first, to get a feel for what I'm asking.)

Answer: We are asked to replace the cost for x_1 by $4+\Delta$. Doing so requires us to recompute the bottom row of the tableau as usual. (We've done several problems like this before, but never with a variable Δ.) When we're done we get the following tableau.

		$4+\Delta$	6	2	0	0	0	
$\underline{c}_B$		x_1	x_2	x_3	u_1	u_2	u_3	
0	u_1	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{3}{4}$	$\frac{7}{4}$
6	x_2	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{9}{4}$
$4+\Delta$	x_1	1	0	1	0	0	1	6
		0	0	$\frac{1}{2}+\Delta$	0	$\frac{3}{2}$	$\frac{5}{2}+\Delta$	$\frac{75}{2}+6\Delta$

Our old solution, $\begin{bmatrix} 6 & \frac{9}{4} & 0 \end{bmatrix}^T$, will remain optimal for this new problem if we don't have to apply the Simplex Method to the tableau above. So we need the objective row of this tableau to remain nonnegative. That means that we need

$$\frac{1}{2} + \Delta \geq 0$$

and

$$\frac{5}{2} + \Delta \geq 0.$$

Therefore the range of Δ for which our old solution remains optimal is $\Delta \geq -1/2$.

2 Given that

$$\underline{x} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}$$

is a solution to the primal problem

$$\begin{array}{l} \text{Maximize } z = x_1 + 4x_2 + 5x_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} -x_1 & + & x_2 & + & x_3 & \leq & 4 \\ 3x_1 & + & x_2 & + & x_3 & \leq & 16 \\ & & x_2 & & & \geq & 1 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \end{array}$$

use the Principle of Complementary Slackness to find a solution to the dual problem

$$\begin{array}{l} \text{Minimize } z = 4w_1 + 16w_2 - w_3 \\ \text{subject to} \\ \begin{array}{rcrcrcl} -w_1 & + & 3w_2 & & & \geq & 1 \\ w_1 & + & w_2 & - & w_3 & \geq & 4 \\ w_1 & + & w_2 & & & \geq & 5 \\ w_1, & & w_2, & & w_3 & \geq & 0 \end{array} \end{array}$$

Answer: First we plug our solution to the primal problem into those constraints to check for slack:

$$\begin{array}{rcrcrcl} -3 & + & 1 & + & 6 & = & 4 \\ 3(3) & + & 1 & + & 6 & = & 16 \\ & & 1 & & & = & 1 \end{array}$$

We didn't find any slack, so the Principle of Complementary Slackness doesn't tell us anything yet. But notice that every entry in our primal problem solution is non-zero, so Complementary Slackness does tell us that there can't be any slack in any of the constraints in the dual problem. So if w_1, w_2, and w_3 represent an optimal solution to the dual problem, we must have

$$\begin{array}{rcrcrcl} -w_1 & + & 3w_2 & & & = & 1 \\ w_1 & + & w_2 & - & w_3 & = & 4 \\ w_1 & + & w_2 & & & = & 5 \end{array}$$

One could solve this using Linear Algebra, but I'll do it the ad hoc way. From the second and third constraints, $w_3 = 1$. Add the first and third constraints together, we see that $4w_2 = 6$, so $w_2 = 3/2$. Then w_1 must be $7/2$. Our solution is therefore

$$\begin{bmatrix} \frac{7}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}.$$

3 Consider the following linear programming problem.

$$\begin{array}{l} \text{Maximize } z = 3x_1 - x_2 + 2x_3 + 4x_4 \\ \text{subject to} \\ \begin{array}{rcrcrcrcr} & & x_2 & + & 7x_3 & + & 2x_4 & \geq & 3, \\ x_1 & + & 2x_2 & + & x_3 & & & = & 9, \\ 2x_1 & + & 3x_2 & + & x_3 & - & 4x_4 & \leq & 7, \end{array} \\ x_j \geq 0,\ j = 1, 2, 3, 4. \end{array}$$

When solving it, the following tableau came up

	x_1	x_2	x_3	x_4	u_1	u_2	
x_4	0	$\frac{1}{4}$	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	$\frac{11}{4}$
u_1	0	$-\frac{1}{2}$	$-\frac{13}{2}$	0	1	$-\frac{1}{2}$	$\frac{5}{2}$
x_1	1	2	1	0	0	0	9
	0	8	2	0	0	16	38

Could it be correct? Explain.

Answer: No. The trick is to recompute the objective row using the rest of the tableau. The mistake is in the u_2 column. In the tableau, this entry is 16, while it should be

$$\begin{bmatrix} 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{2} \\ 0 \end{bmatrix} - 0 = -1.$$