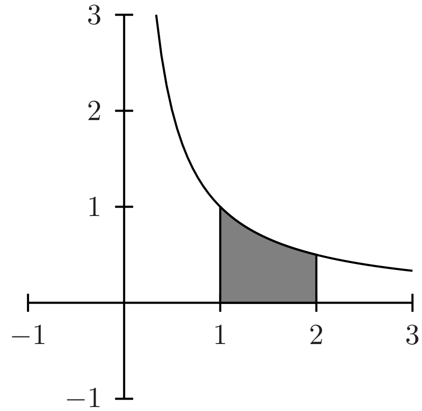
1 Let $\mathcal{R}$ denote the region bounded by $y = \dfrac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$.

(a) Find the volume of the solid that results from rotating $\mathcal{R}$ around the x-axis.

(b) Find the volume of the solid that results from rotating $\mathcal{R}$ around the y-axis.

Solution: The region $\mathcal{R}$ is shown below:



We can find the volume of the solid formed by rotating $\mathcal{R}$ around the x-axis using discs:

$$\begin{aligned} \text{Volume} &= \int_1^2 \pi \left(\frac{1}{x}\right)^2 \, dx \\ &= \pi \int_1^2 x^{-2} \, dx \\ &= \pi \left[-x^{-1}\right]_1^2 \\ &= \frac{\pi}{2}. \end{aligned}$$

We could use washers to find the volume when rotated about the y-axis, and then we'd be integrating with respect to y. Instead, I'll use shells. The height of one of our shells is $1/x$ and the radius is x, so we get

$$\begin{aligned} \text{Volume} &= \int_1^2 2\pi x \left(\frac{1}{x}\right)^2 \, dx \\ &= 2\pi \int_1^2 \frac{1}{x} \, dx \\ &= 2\pi \left[\ln x\right]_1^2 \\ &= 2\pi \ln 2. \end{aligned}$$

2 Determine whether each of the following improper integrals converge or diverge. Calculate the value of any convergent integral.

$$\int_0^\infty xe^{-3x^2}\ dx \qquad\qquad \int_{-2}^0 \frac{dx}{2+x}$$

Solution: We can integrate $\int xe^{-3x^2}\ dx$ by making a u-substitution:

$$\begin{aligned} u &= -3x^2 \\ du &= -6x\ dx \\ dx &= \frac{du}{-6x} \end{aligned}$$

Then we get

$$\begin{aligned} \int_0^\infty xe^{-3x^2}\ dx &= \lim_{t\to\infty}\int_0^t xe^{-3x^2}\ dx \\ &= \lim_{t\to\infty}\int_{x=0}^{x=t} \frac{e^u}{-6}\ du \\ &= \lim_{t\to\infty}\left[\frac{e^u}{-6}\right]_{x=0}^{x=t} \\ &= \lim_{t\to\infty}\left[\frac{e^{-3x^2}}{-6}\right]_0^t \\ &= \lim_{t\to\infty}\frac{e^{-3t^2}}{-6}+\frac{1}{6} \\ &= \frac{1}{6}. \end{aligned}$$

So this integral is convergent.

For the other integral, note that $\frac{dx}{2+x}$ has a pole as $x=-2$, so the proper form of the integral is

$$\lim_{t\to-2}\int_t^0 \frac{dx}{2+x}.$$

This integral is just as easy to integral, we just set $u=2+x$, so $du=dx$, and we get

$$\begin{aligned} \lim_{t\to-2}\int_t^0 \frac{dx}{2+x} &= \lim_{t\to-2}\int_{x=t}^{x=0}\frac{du}{u} \\ &= \lim_{t\to-2}\left[\ln(u)\right]_{x=t}^{x=0} \\ &= \lim_{t\to-2}\left[\ln(x+2)\right]_t^0 \\ &= \ln(2)-\lim_{t\to-2}\ln(t+2). \end{aligned}$$

Note that as t approaches -2, $t+2$ approaches 0, and thus $\ln(t+2)$ tends to $-\infty$. Therefore this integral is divergent.

3 Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{y^2(x^2+2x+3)}{(x^2+1)(x+1)}$$

satisfying the initial condition $y(0) = 2$. In the answer express y as a function of x.

Solution: This is a separable equation, and the first thing we do is separate it:

$$\frac{dy}{y^2} = \frac{x^2+2x+3}{(x^2+1)(x+1)}\,dx.$$

Now we want to integrate both sides. The left-hand side is easy:

$$\int \frac{dy}{y^2} = -\frac{1}{y}.$$

Integrating the right-hand side requires partial fractions. Write

$$\frac{x^2+2x+3}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Cancel denominators:

$$x^2+2x+3 = A(x^2+1) + (Bx+C)(x+1),$$

and expand to get

$$x^2+2x+3 = (A+B)x^2 + (B+C)x + (A+C).$$

This gives us the system of equations

$$\begin{cases} A+B &= 1 \\ B+C &= 2 \\ A+C &= 3 \end{cases}$$

We might as well start at the bottom; since $A+C=3$, $C=3-A$. Now we can plug that into the second equation, $B+C=2$, to get $B+(3-A)=2$, so $B-A=-1$. Adding this to the top equation gives $B=0$. Now we can work backwards: since $B-A=-1$ and $B=0$, $A=1$, and then $C=3-A=3-1=2$. This gives us

$$\int \frac{x^2+2x+3}{(x^2+1)(x+1)}\,dx = \int \frac{1}{x+1} + \frac{2}{x^2+1}\,dx = \ln(x+1) + 2\arctan(x) + D,$$

where D is the constant of integration. Therefore we have

$$-\frac{1}{y} = \ln(x+1) + 2\arctan(x) + D,$$

and solving for y gives

$$y = \frac{-1}{\ln(x+1) + 2\arctan(x) + D}.$$

Finally, we have to plug $x = 0$ in to get

$$2 = y(0) = \frac{-1}{\ln(0+1) + 2\arctan(0) + D} = \frac{-1}{D},$$

so we need to set $D = -1/2$ to get

$$y = \frac{-1}{\ln(x+1) + 2\arctan(x) - \frac{1}{2}}.$$

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4 Let $f(x) = \dfrac{3x}{2+5x^4}$.

(a) Find the power series representation for $f(x)$.

(b) What is the radius of convergence of this series?

Solution: We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

so we would like to get out function into this form. First let's pull the $3x$ out of the numerator and a 2 out of the denominator to get

$$\frac{3x}{2+5x^4} = \frac{3x}{2} \cdot \frac{1}{1+\frac{5}{2}x^4}.$$

We then have

$$\frac{1}{1+\frac{5}{2}x^4} = \frac{1}{1-\left(-\frac{5}{2}x^4\right)} = \sum_{n=0}^{\infty} \left(-\frac{5}{2}x^4\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{5}{2}\right)^n x^{4n}.$$

Now we just put the $\dfrac{3x}{2}$ back in:

$$\frac{3x}{2+5x^4} = \frac{3x}{2} \cdot \frac{1}{1+\frac{5}{2}x^4} = \frac{3x}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{5}{2}\right)^n x^{4n} = \sum_{n=0}^{\infty} (-1)^n \frac{3}{2}\left(\frac{5}{2}\right)^n x^{4n+1}.$$

To find the radius of convergence, we can use the Root Test. The relevant limit is

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{3}{2}\left(\frac{5}{2}\right)^n x^{4n+1}\right|} &= \left(\lim_{n\to\infty} \sqrt[n]{\frac{3}{2}}\right)\left(\lim_{n\to\infty} \sqrt[n]{\left(\frac{5}{2}\right)^n}\right)\left(\lim_{n\to\infty} \sqrt[n]{|x^{4n+1}|}\right) \\
&= 1 \cdot \frac{5}{2} \cdot \lim_{n\to\infty} \left|x^{4+\frac{1}{n}}\right| \\
&= \frac{5}{2}\left|x^4\right|.
\end{aligned}$$

Since we need this limit to be less than 1, we solve for

$$\frac{5}{2}\left|x^4\right| < 1.$$

This gives

$$|x| < \sqrt[4]{\frac{2}{5}}.$$

Therefore the radius of convergence is $\sqrt[4]{\frac{2}{5}}$. Note that if we were asked for the interval of convergence, we would have to check the endpoints $-\sqrt[4]{\frac{2}{5}}$ and $\sqrt[4]{\frac{2}{5}}$.

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5 Let k be a fixed integer less than 8. Show that

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx = \frac{1}{2^{k/2}}\sum_{n=0}^{\infty}\frac{1}{16^n(8n+k)}.$$

Solution: The power series for $\frac{1}{1-x^8}$ is $\sum_{n=0}^{\infty} x^{8n}$, so

$$\frac{x^{k-1}}{1-x^8} = \sum_{n=0}^{\infty} x^{8n+k-1}.$$

Thus

$$\begin{aligned}
\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx &= \int_0^{1/\sqrt{2}} \sum_{n=0}^{\infty} x^{8n+k-1} \\
&= \sum_{n=0}^{\infty} \int_0^{1/\sqrt{2}} x^{8n+k-1} \\
&= \sum_{n=0}^{\infty} \left[\frac{x^{8n+k}}{8n+k}\right]_0^{1/\sqrt{2}} \\
&= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{\sqrt{2}}\right)^{8n+k}}{8n+k} \\
&= \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \cdot \frac{1}{2^{k/2}} \cdot \frac{1}{8n+k}.
\end{aligned}$$

Since the $\frac{1}{2^{k/2}}$ does not depend on n, we can pull it out of the sum, and we can rewrite $\frac{1}{2^{4n}}$ as $\frac{1}{16^n}$ to get

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx = \frac{1}{2^{k/2}}\sum_{n=0}^{\infty}\frac{1}{16^n(8n+k)}.$$

6 Show that

$$\int_0^{1\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1 - x^8} = \pi.$$

Hint: Once we substitute $u = \sqrt{2}x$, the integral becomes

$$\int_0^1 \frac{16u - 16}{u^4 - 2u^3 + 4u - 4}\ du,$$

and using partial fractions, this is equal to

$$\int_0^1 \frac{4u}{u^2 - 2}\ du - \int_0^1 \frac{4u - 8}{u^2 - 2u + 2}\ du.$$

Solution: First let's figure out what $\int_0^1 \frac{4u}{u^2 - 2}\ du$ is. If we set $w = u^2 - 2$ then $dw = 2u\ du$ so $du = dw/2u$, and we have

$$\begin{aligned} \int_0^1 \frac{4u}{u^2 - 2}\ du &= \int_{-2}^{-1} \frac{2}{w}\ dw \\ &= [2\ln|w|]_{-2}^{-1} \\ &= 2\ln(1) - 2\ln(2) \\ &= -2\ln(2). \end{aligned}$$

Now we have to do something with $\int_0^1 \frac{4u - 8}{u^2 - 2u + 2}\ du$. Note that setting $w = u^2 - 2u + 2$ would leave us with $dw = 2u - 2\ du$, and so it's not going to work.

Instead we need a trig substitution. We start by writing the denominator as $(something)^2 + somethingelse$:

$$u^2 - 2u + 2 = (u - 1) + 1.$$

So now we have

$$\int_0^1 \frac{4u - 8}{u^2 - 2u + 2}\ du = \int_0^1 \frac{4(u - 1) - 4}{(u - 1)^2 + 1}\ du.$$

So our substitution is

$$\begin{aligned} u - 1 &= \tan\theta \\ du &= \sec^2\theta\ d\theta \end{aligned}$$

and after plugging this in we get

$$\begin{aligned} \int_0^1 \frac{4(u - 1) - 4}{(u - 1)^2 + 1}\ du &= \int_{u=0}^{u=1} \frac{4\tan\theta - 4}{\tan^2\theta + 1}\ \sec^2\theta\ d\theta \\ &= \int_{u=0}^{u=1} \frac{4\tan\theta - 4}{\sec^2\theta}\ \sec^2\theta\ d\theta \\ &= \int_{u=0}^{u=1} 4\tan\theta - 4\ d\theta \\ &= 4\left[\ln|\sec\theta| + \theta\right]_{u=0}^{u=1}. \end{aligned}$$

We want to get back to having us in the formula, so we have to figure out what $\sec\theta$ is. For this we go back to our right-triangle. Since we set $\tan\theta = u-1$, we can say that the opposite side is $u-1$ and the adjacent side is 1. By the Pythagorean theorem, the hypotenuse is then $\sqrt{(u-1)^2+1}$. So $\sec\theta = \dfrac{\textit{hypotenuse}}{\textit{adjacent}} = \sqrt{(u-1)^2+1}$. Also, θ is clearly $\arctan(u-1)$. This gives us

$$4\left[\ln\left|\sqrt{(u-1)^2+1}\right| + \arctan(u-1)\right]_{u=0}^{u=1} = -2\ln(2) - \pi.$$

Putting this all together we get

$$\int_0^1 \frac{4u}{u^2-2}\,du - \int_0^1 \frac{4u-8}{u^2-2u+2}\,du = (-2\ln(2)) - (-2\ln(2)-\pi) = \pi,$$

as desired.

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7 Using the previous two problems, show that

$$\sum_{n=0}^{\infty} \frac{1}{16^n}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right) = \pi,$$

thereby proving the famous Bailey-Borwein-Plouffe formula for π, discovered in 1995. (Therefore, if you had done this problem ten years ago, you would be famous.)[1]

Solution: We know that

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx = \frac{1}{2^{k/2}} \sum_{n=0}^{\infty} \frac{1}{16^n(8n+k)},$$

or in other words,

$$\int_0^{1/\sqrt{2}} \frac{2^{k/2}\cdot x^{k-1}}{1-x^8}\,dx = \sum_{n=0}^{\infty} \frac{1}{16^n(8n+k)},$$

for any fixed $k < 8$. So:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{16^n}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right) &= \int_0^{1/\sqrt{2}} \frac{4\cdot 2^{1/2}}{1-x^8} - \frac{2\cdot 2^{4/2}\cdot x^3}{1-x^8} - \frac{2^{5/2}\cdot x^4}{1-x^8} - \frac{2^{6/2}\cdot x^5}{1-x^8} \\ &= \int_0^{1\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8},\end{aligned}$$

and we know that this is π.

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[1] To see how the formula was actually discovered, read their original paper, "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation* **66** (1997), 903–913 or the book *Mathematics by Experiment* by Borwein and Bailey