$\boxed{1}$ Determine whether $\lim_{n\to\infty} a_n$ exists when the sequence $\{a_n\}$ is defined as follows. Find the limit if it exists.

$$a_n = \frac{(n+2)!}{n!(3+5n)^2} \qquad a_n = \frac{5n}{\ln(2+3e^n)} \qquad a_n = (2n)^{1/n}$$

Solution: Since

$$\begin{aligned}(n+2)! &= (n+2)\cdot(n+1)\cdot n\cdot\ldots\cdot 2\cdot 1\\ &= (n+2)(n+1)\cdot n!\end{aligned}$$

so the first sequence simplifies to

$$a_n = \frac{(n+2)!}{n!(3+5n)^2} = \frac{(n+2)(n+1)}{(3+5n)^2}.$$

Now expand the numerator and denominator:

$$a_n = \frac{(n+2)(n+1)}{(3+5n)^2} = \frac{n^2+3n+2}{25n^2+30n+9},$$

and pull out the highest power of n:

$$a_n = \frac{n^2}{n^2}\cdot\frac{1+\frac{3}{n}+\frac{2}{n^2}}{25+\frac{30}{n}+\frac{9}{n^2}}.$$

The limit is now easy to spot:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1+\frac{3}{n}+\frac{2}{n^2}}{25+\frac{30}{n}+\frac{9}{n^2}} = \frac{1}{25}.$$

$\dfrac{5n}{\ln(2+3e^n)}$: This can be handled with L'Hospital's Rule, since it is of the form $\frac{\infty}{\infty}$:

$$\lim_{n\to\infty}\frac{5n}{\ln(2+3e^n)} = \lim_{n\to\infty}\frac{(5n)'}{(\ln(2+3e^n))'} = \lim_{n\to\infty}\frac{5}{\left(\frac{1}{2+3e^n}\right)\cdot 3e^n}.$$

We just need to continue manipulating this fraction:

$$\lim_{n\to\infty}\frac{5(2+3e^n)}{3e^n} = \lim_{n\to\infty}\frac{10}{3e^n}+\frac{15e^n}{3e^n} = \lim_{n\to\infty}\frac{10}{3e^n}+5 = 5.$$

$(2n)^{1/n}$: Since there is an n in the exponent, we first want to rewrite it in the form $e^{\text{something}}$:

$$a_n = (2n)^{1/n} = \left(e^{\ln 2n}\right)^{1/n} = e^{\frac{\ln 2n}{n}}.$$

Now:

$$\lim_{n\to\infty} e^{\frac{\ln 2n}{n}} = e^{\left(\lim_{n\to\infty} \frac{\ln 2n}{n}\right)}.$$

To figure out what $\lim\limits_{n\to\infty} \dfrac{\ln 2n}{n}$ is we can use L'Hospital's Rule again:

$$\lim_{n\to\infty} \frac{\ln 2n}{n} = \lim_{n\to\infty} \frac{(\ln 2n)'}{(n)'} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

So we get

$$\lim_{n\to\infty} a_n = e^0 = 1.$$

Alternatively, if we take $\lim\limits_{n\to\infty} n^{1/n} = 1$ as a known fact then we can just write $(2n)^{1/n}$ as $2^{1/n} \cdot n^{1/n}$, from which it follows by the rules of limits that

$$\lim_{n\to\infty} (2n)^{1/n} = \left(\lim_{n\to\infty} 2^{1/n}\right)\left(\lim_{n\to\infty} n^{1/n}\right) = 1 \cdot 1 = 1.$$

$\boxed{2}$ Determine whether the following series converge. Sum the series that do converge.

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{5^n} \qquad \sum_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{n^2}\right) \qquad \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

Solution: The first series can be written as a sum of two geometric series:

$$\sum_{n=1}^{\infty} \frac{3^n + 4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{5^n} + \frac{4^n}{5^n}\right) = \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n.$$

The only tricky part is that these series start at $n = 1$, not $n = 0$. Once you notice this, you get

$$\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n = \frac{3/5}{1 - 3/5} + \frac{4/5}{1 - 4/5}.$$

$\sum\limits_{n=1}^{\infty} \left(1 + \dfrac{(-1)^n}{n^2}\right)$: This series diverges because it's terms do not tend to 0 (the book calls this the Test for Divergence):

$$\lim_{n\to\infty} \left(1 + \frac{(-1)^n}{n^2}\right) = 1 + \lim_{n\to\infty} \frac{(-1)^n}{n^2} = 1.$$

$\sum\limits_{n=1}^{\infty} \dfrac{2n+1}{n^2(n+1)^2}$: This series converges by limit comparison to $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$. Finding the sum is is trickier, and requires partial fractions. We can express the summand as

$$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

Now we can plug this back into the series to get

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

This is a telescoping series, and it's probably easiest to see the answer if we first break it up into two series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

Now notice that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \\ \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} &= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \end{aligned}$$

So all the terms except 1 cancel each other out, leaving

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1.$$

Solution: For which values of p does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? For which values of p does $\sum_{n=1}^{\infty} \frac{1}{p^n}$ converge?

Solution: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$. This is precisely the p-Test, which can be proved by comparing the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ to the indefinite integral $\int_1^{\infty} x^{-p}\, dx$.

$\sum_{n=1}^{\infty} \frac{1}{p^n}$ converges when $|p| > 1$. This can we shown in a variety of ways. First, you could use the Root Test, which would say that the series converges if

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{p^n}\right|} = \left|\frac{1}{p}\right| < 1,$$

from which it follows that the series converges when $|p| > 1$, and then you just have to check the endpoints: $p = 1$ and $p = -1$ (the series diverges at both of them).

Or you could use the Ratio Test, which produce the same inequality, $|1/p| < 1$.

But the easiest way is to notice that this series is geometric with ratio $1/p$:

$$\sum_{n=1}^{\infty} \frac{1}{p^n} = \sum_{n=1}^{\infty} \left(\frac{1}{p}\right)^n,$$

so from what we know about geometric series, it converges if $|1/p| < 1$, i.e., if $|p| > 1$.

$\boxed{3}$ Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=0}^{\infty} \frac{n^3}{3^n} \qquad \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \qquad \sum_{n=1}^{\infty} \frac{n^3}{n^4+1} \qquad \sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^4+1}$$

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n n}{\sqrt{n^2+1}} \qquad \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$$

Solution: There are a lot of series here, so only sketches will be given.

$\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{3^n}$: Convergent. The Ratio Test could be used, and then you would want to show that

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \cdot \frac{1}{3} \right| < 1,$$

which it most certainly is.

$\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$: Absolutely convergent, by comparison to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ (which converges by the p-Test).

$\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$: Divergent. You could show this using the Integral Test because

$$\int_1^{\infty} \frac{x^3}{x^4+1}\, dx = \infty,$$

but to make this rigorous you would also need to verify that $\frac{x^3}{x^4+1}$ is decreasing.

A much easier way is to use limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^4+1}$: Convergent. Use limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{7/2}}$, which converges by the p-Test. In fact, since $n^4+1 \geq n^4$,

$$\frac{\sqrt{n}}{n^4+1} \leq \frac{\sqrt{n}}{n^4} = \frac{1}{n^{7/2}},$$

so

$$\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^4+1} \leq \sum_{n=0}^{\infty} \frac{1}{n^{7/2}},$$

and we don't even need to use limit comparison.

$\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$: Divergent by the Integral Test because

$$\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx$$

is divergent. (To evaluate this integral let $u = \ln x$.)

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}$: Conditionally convergent. The series converges by the Alternating Series Test, but it is not absolutely convergent because

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^{1/3}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$$

diverges by the p-Test.

$\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n}{\sqrt{n^2+1}}$: Divergent. The limit of the summands does not tend to 0, so the series automatically diverges:

$$\lim_{n\to\infty}\left|\frac{(-1)^n n}{\sqrt{n^2+1}}\right| = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty}\frac{n}{n}\cdot\frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1.$$

$\displaystyle\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}$: Because this series has a factorial (the !), we use the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)!}{n!}\right)^2\cdot\frac{(2n)!}{(2n+2)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4}.$$

$\boxed{4}$ We wish to approximate the sum S of the series $\displaystyle\sum_{n=1}^{\infty}a_n$ with the partial sum $s_N = \displaystyle\sum_{n=1}^{N}a_n$. How large should N be to ensure that s_N approximates S with an error less that 10^{-4} for each of the following series?

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad\qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

Solution: For the first series, we have to compare the remainder to an integral. Since we want to find an N so that $\displaystyle\sum_{n=1}^{N}\frac{1}{n^2}$ is with 10^{-4} of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$, so we want to find an N so that

$$\sum_{n=N+1}^{\infty}\frac{1}{n^2} \le 10^{-4}.$$

Then using the Integral Test we get

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{1}{x^2}\, dx = \frac{1}{N},$$

so we want

$$\frac{1}{N} \leq 10^{-4},$$

and thus $N \geq 10^4$ will work.

The second series is alternating, so it's easier to analyze. The difference between the partial sum $\sum_{n=0}^{N} \frac{(-1)^n}{n^2}$ and the infinite sum $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is at most the absolute value of $N+1^{\text{st}}$ term of the series, $\frac{1}{(N+1)^2}$. So we want

$$\frac{1}{(N+1)^2} \leq 10^{-4},$$

and solving this gives $N \geq 99$.

$\boxed{5}$ Find the radius of convergence of the following power series and determine the convergence behavior at the endpoints of the interval of convergence. Show your reasoning.

$$\sum_{n=1}^{\infty} \frac{n^3 x^n}{3^n} \qquad \sum_{n=1}^{\infty} \frac{5^n x^n}{n^{1/4}} \qquad \sum_{n=1}^{\infty} \frac{n^2}{(3n+1)!} x^n$$

Solution: For the first power series you can use either the Root Test or the Ratio Test. The Root Test shows that the series converges if

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^3 x^n}{3^n}\right|} = \lim_{n\to\infty} \sqrt[n]{n^3} \left|\frac{x}{3}\right| = \left|\frac{x}{3}\right| < 1,$$

so the radius of convergence is 3.

If you use the Ratio Test, you'll see that the series converges if

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^3 x^{n+1}}{3^{n+1}}}{\frac{n^3 x^n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^3}{n^3} \cdot \frac{x}{3} \right| = \left|\frac{x}{3}\right| < 1,$$

the same conclusion.

Now we need to check the endpoints, $x = 3$ and $x = -3$. When you plug in $x = 3$ you get

$$\sum_{n=1}^{\infty} n^3,$$

which diverges because the summands are not tending to 0 (n^3 heads off to ∞). Similarly, when you plug in $x = -3$ you get

$$\sum_{n=1}^{\infty} (-1)^n n^3,$$

and this diverges for the same reason.

Therefore the interval of convergence is $-3 < x < 3$ or $(-3, 3)$.

$\displaystyle\sum_{n=1}^{\infty} \frac{5^n x^n}{n^{1/4}}$: Like the last problem, you can use either the Root Test or the Ratio Test to find the radius of convergence. If you use the Root Test, you'll need to have

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{5^n x^n}{n^{1/4}}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^{1/4}}}\, |5x| = \lim_{n\to\infty} \left(n^{-1/4n}\right) |5x| = |5x| < 1.$$

(To see that $\lim_{n\to\infty} n^{-1/4n} = 1$, you might want to write it as $\lim_{n\to\infty} \left(n^{1/n}\right)^{-1/4} = \left(\lim_{n\to\infty} n^{1/n}\right)^{-1/4}$. We've run across this limit a few times already; $\lim_{n\to\infty} n^{1/n} = 1$.)

Thus the radius of convergence is $1/5$, and now we need to check the endpoints.

Plugging in $x = 1/5$ gives us the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/4}},$$

which diverges by the p-Test.

Plugging in $x = -1/5$ gives

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/4}},$$

which converges by the Alternating Series Test.

Therefore the interval of convergence is $-\frac{1}{5} \le x < \frac{1}{5}$, or $[-1/5, 1/5)$.

$\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(3n+1)!} x^n$: Since this series involves a factorial, we should use the Ratio Test. It says that the series will converge if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2}{(3(n+1)+1)!} x^{n+1}}{\frac{n^2}{(3n+1)!} x^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2} \cdot \frac{(3n+1)!}{(3n+4)!} \cdot x\right| < 1.$$

Now notice that $\dfrac{(3n+1)!}{(3n+4)!} = \dfrac{1}{(3n+4)(3n+3)(3n+2)}$ and $\lim_{n\to\infty} \dfrac{(n+1)^2}{n^2} = 1$, so we need

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x}{(3n+4)(3n+3)(3n+2)}\right| < 1.$$

Since this limit is 0 for any x, the series will always converge, so it's radius of convergence is ∞ and the interval of convergence is $-\infty < x < \infty$ or $(-\infty, \infty)$.

$\boxed{6}$ By manipulating series that you already know, find Maclaurin series for each of the following functions.

$$\frac{1}{1+x^3} \qquad x\sin 2x \qquad \frac{\ln(1+x^2)}{x}$$

$$\frac{e^x-e^{-x}}{2} \qquad e^{3x^2} \qquad -\left(\frac{1}{1+x}\right)^2$$

Solution: First, here are some Maclaurin series that you should know:

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}
\end{aligned}$$

You should also either remember the Maclaurin series for $\ln(1+x)$ or remember how to derive it by writing $\ln(1+x)$ as the integral of $\frac{1}{1+x}$, which has Maclaurin series $\sum_{n=0}^{\infty}(-1)x^n$:

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^n\,dx = \sum_{n=0}^{\infty}(-1)^n \int x^n\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

Moving on to the functions in the problem...

$\frac{1}{1+x^3}$: We put this in the form $\frac{1}{1-\text{something}}$ and then use the Maclaurin series for $\frac{1}{1-x}$:

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n}.$$

$x\sin 2x$: First get the series for $\sin 2x$:

$$\sin 2x = \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{2n+1}}{(2n+1)!}.$$

Now multiply the x in:

$$x\sin 2x = x\left(\sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} x(-1)^n \frac{2^{2n+1}x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}x^{2n+2}}{(2n+1)!}.$$

$\dfrac{\ln(1+x^2)}{x}$: First get the series for $\ln(1+x^2)$ using the series for $\ln(1+x)$ that we derived above:

$$\ln(1+x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1}.$$

Now divide by x:

$$\frac{\ln(1+x^2)}{x} = \frac{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1}}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{\frac{x^{2n+2}}{n+1}}{x} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{n+1}.$$

$\dfrac{e^x - e^{-x}}{2}$: We already have the series for e^x, and the series for e^{-x} is

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}.$$

Putting this together:

$$e^x + e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}.$$

Now you want to collect coefficients of x^n:

$$e^x - e^{-x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}. = \sum_{n=0}^{\infty} \left(1-(-1)^n\right) \frac{x^n}{n!}.$$

To simplify this further, notice that $1-(-1)^n$ is 0 if n is even and 2 if n is odd. So we can rewrite the series to only sum over odd integers:

$$e^x - e^{-x} = \sum_{m=0}^{\infty} 2\frac{x^{2m+1}}{(2m+1)!}.$$

Finally we divide by 2:

$$\frac{e^x - e^{-x}}{2} = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}.$$

Note that this is the series for $\sin x$, except without the $(-1)^n$. This function is hyperbolic sine, written $\sinh x$.

e^{3x^2}: We just have to plug $3x^2$ into the series for e^x:

$$e^{3x^2} = \sum_{n=0}^{\infty} \frac{\left(3x^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}.$$

$-\left(\dfrac{1}{1+x}\right)^2$: The easiest way to find this series is to recognize that

$$-\left(\frac{1}{1+x}\right)^2 = \frac{d}{dx}\left(\frac{1}{1+x}\right).$$

Now since we know that $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$, we have that

$$-\left(\frac{1}{1+x}\right)^2 = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right) = \sum_{n=0}^{\infty}(-1)^n \frac{d}{dx}\left(x^n\right) = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}.$$

$\boxed{7}$ Find the first few terms of the Maclaurin series for the following functions.

$$(\ln(1+x))^2 \qquad \frac{\cos x}{1-x} \qquad \sin^2 x + \cos^2 x$$

Solution: For all of these, you could use Taylor's Theorem:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(with $a = 0$, since the problem asked for Maclaurin series), but that's almost never the most efficient method by hand. Instead you should try to express the functions as products of functions that you know the series of.

$(\ln(1+x))^2$: We know the series for $\ln(1+x)$, so this is

$$(\ln(1+x))^2 = \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}\right)^2 = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots\right)^2.$$

Now just square the thing to get

$$(\ln(1+x))^2 = x^2 - x^3 + \frac{11}{12}x^4 - \dots$$

$\frac{\cos x}{1-x}$: This is $\cos x$ times $\frac{1}{1-x}$, and we know the series for both:

$$\begin{aligned}
\frac{\cos x}{1-x} &= \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right)\left(\sum_{n=0}^{\infty} x^n\right) \\
&= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \dots\right)\left(1 + x + x^2 + x^3 + x^4 + \dots\right) \\
&= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^4 + \dots
\end{aligned}$$

$\sin^2 x + \cos^2 x$: There is certainly a quick way to find the Maclaurin series for this function, but let's pretend we don't see it. Then we have

$$\sin^2 x = \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)^2 = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots\right)^2 = x^2 - \frac{1}{3}x^4 + \dots,$$

and

$$\cos^2 x = \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\right)^2 = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots\right)^2 = 1 - x^2 + \frac{1}{3}x^4 - \dots$$

So when we add these together:

$$\sin^2 x + \cos^2 x = \left(x^2 - \frac{1}{3}x^4 + \dots\right) + \left(1 - x^2 + \frac{1}{3}x^4 - \dots\right) = 1.$$

$\boxed{8}$ Use the Maclaurin series for e^x to obtain a power series for the function

$$f(x) = \int_0^x \frac{1 - e^{t^2}}{t}\, dt.$$

Solution: We will work from the inside out. The Maclaurin series for e^t (which you should know) is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!},$$

so to get the series for e^{t^2} we substitute t^2 into this series:

$$e^{t^2} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!}.$$

Now we want the series for $1 - e^{t^2}$. First pull off the constant term for e^{t^2}:

$$e^{t^2} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{t^{2n}}{n!}.$$

So

$$1 - e^{t^2} = 1 - \left(1 + \sum_{n=1}^{\infty} \frac{t^{2n}}{n!}\right) = -\sum_{n=1}^{\infty} \frac{t^{2n}}{n!}.$$

Now we need to divide by t:

$$\frac{1 - e^{t^2}}{t} = \frac{-\displaystyle\sum_{n=1}^{\infty} \frac{t^{2n}}{n!}}{t} = -\sum_{n=1}^{\infty} \frac{t^{2n}}{n!} \cdot \frac{1}{t} = -\sum_{n=1}^{\infty} \frac{t^{2n-1}}{n!}.$$

Now we are finally ready to integrate, which we do term-by-term:

$$\int -\sum_{n=1}^{\infty} \frac{t^{2n-1}}{n!}\, dt = -\sum_{n=1}^{\infty} \int \frac{t^{2n-1}}{n!}\, dt = -\sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)n!}.$$

Finally we evaluate the integral to get

$$\int_0^x \frac{1 - e^{t^2}}{t}\, dt = -\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)n!}.$$

$\boxed{9}$ Find $\lim_{x\to 0} \frac{(\sin x - x)^2}{(\cos 5x - 1)^3}$. **Solution:** This is not a good problem to try L'Hospital's Rule on. Instead you want to do this with power series. Recall that

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\\ \cos x &= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

So by picking off the x term in the series for $\sin x$ we get

$$\sin x - x = \left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) - x = \left(x + \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) - x = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

Similarly

$$\cos 5x - 1 = \left(\sum_{n=0}^{\infty}(-1)^n \frac{5^{2n}x^{2n}}{(2n)!}\right) - 1 = \left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{5^{2n}x^{2n}}{(2n)!}\right) - 1 = \sum_{n=1}^{\infty}(-1)^n \frac{5^{2n}x^{2n}}{(2n)!}.$$

Now we substitute these series into the fraction and square and cube as required:

$$\begin{aligned}\lim_{x\to 0} \frac{(\sin x - x)^2}{(\cos 5x - 1)^3} &= \lim_{x\to 0} \frac{\left(\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)^2}{\left(\sum_{n=1}^{\infty}(-1)^n \frac{5^{2n}x^{2n}}{(2n)!}\right)^3}\\ &= \lim_{x\to 0} \frac{\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)^2}{\left(-\frac{5^2x^2}{2!} + \frac{5^4x^4}{4!} - \frac{5^6x^6}{6!} + \ldots\right)^3}\\ &= \lim_{x\to 0} \frac{\left(-\frac{x^3}{3!}\right)^2 + 2\left(-\frac{x^3}{3!}\right)\left(\frac{x^5}{5!}\right) + \ldots}{\left(-\frac{5^2x^2}{2!}\right)^3 + 3\left(-\frac{5^2x^2}{2!}\right)\left(\frac{5^4x^4}{4!}\right)^2 + 3\left(-\frac{5^2x^2}{2!}\right)^2\left(\frac{5^4x^4}{4!}\right) + \ldots}\end{aligned}$$

We've actually done more work than necessary here, because only the leading terms of the numerator and denominator matter. Therefore our limit is

$$\lim_{x\to 0} \frac{\left(-\frac{x^3}{3!}\right)^2}{\left(-\frac{5^2x^2}{2!}\right)^3} = \lim_{x\to 0} \frac{\frac{x^6}{36}}{-\frac{15625x^6}{8}} = -\frac{8}{36\cdot 15625} = -0.000014222\ldots$$

Solution: Find the Maclaurin series for $\arctan x$ and use this to write π as the sum of an infinite series.

Solution: Since $\arctan x$ is the integral of $\frac{1}{1+x^2}$ we have

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} \left(-x^2\right)^n\,dx = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Now remember that $\arctan(1) = \frac{\pi}{4}$, so

$$\arctan(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4},$$

and thus

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

This is known as the Gregory series. The only problem with this series is that it converges extremely slowly; you need to add up 300 terms of the series to approximate π to within two decimal places. To get a series that converges more quickly, you can use the fact that

$$\pi = 16\arctan\left(\frac{1}{5}\right) - 4\arctan\left(\frac{1}{239}\right),$$

so

$$\pi = 16\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{5}\right)^{2n+1} - 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{1}{239}\right)^{2n+1}.$$

This is known as Machin's formula.