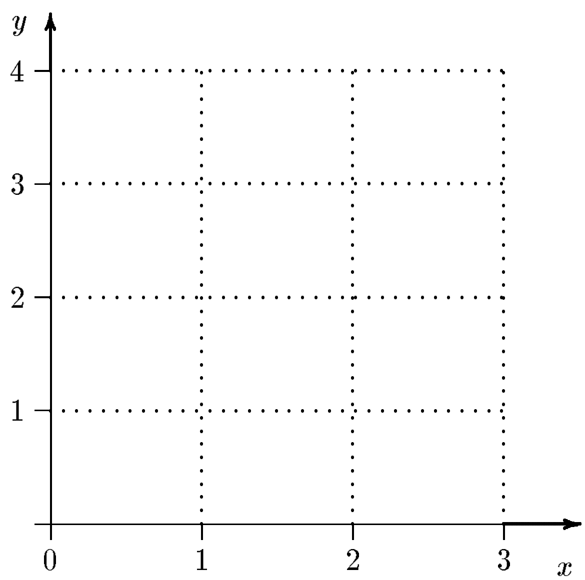
640:152/154 Calculus Final Exam May 7, 1997

(20) 1. a) Verify using calculus that $\displaystyle\int_1^{e^2} \frac{(\ln x)^3}{x}\,dx = 4$

b) Verify using calculus that $\displaystyle\int_0^{\pi/2} x \cos x\,dx = \frac{\pi}{2} - 1$

(16) 2. a) Sketch the region in the first quadrant bounded by the curves $y = x^2$ and $y = 3x(2-x)$. Label the curves and use algebra to find the coordinates of the intersection points.



b) Set up and compute a definite integral for the area of the region sketched in a). Compute this integral using calculus. YOU NEED NOT SIMPLIFY THE NUMERICAL ANSWER.

c) Set up a definite integral for the volume obtained when the region sketched in a) is revolved about the x-axis. Describe the method you are using. DO NOT EVALUATE THE INTEGRAL.

(20) 3. Calculate the following indefinite integrals. Show your work and express your answers in terms of explicit functions of x.

(a) $\displaystyle\int \frac{\sin x}{(\cos x)^3}\,dx$

(b) $\displaystyle\int \sqrt{x}\, e^{3\sqrt{x}}\,dx$

(12) 4. Use the method of partial fractions and calculus to show that $\displaystyle\int_2^5 \frac{4}{(x+1)(x+3)}\,dx = \ln\left(\frac{25}{16}\right)$

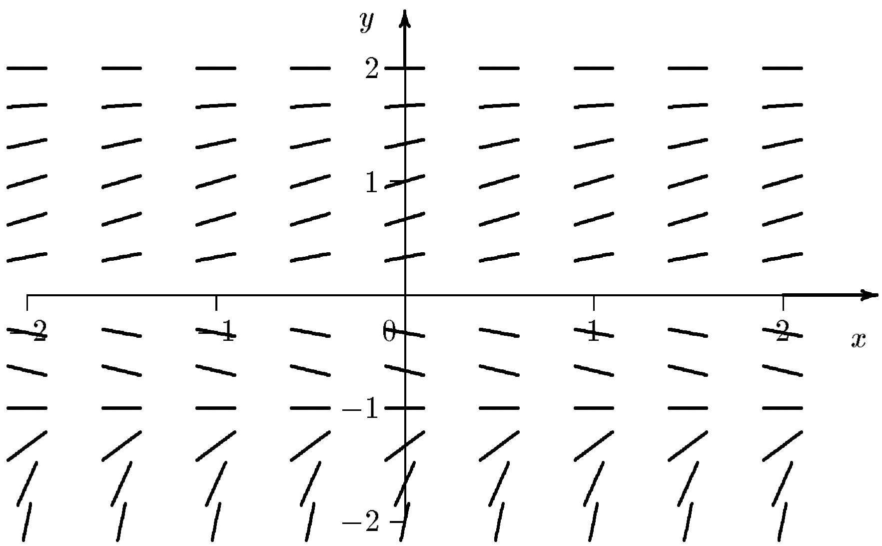
(12) 5. Find the solution to the differential equation $\dfrac{dy}{dx} = 4y\tan x$ that satisfies the initial condition $y(0) = 2$. Express your answer as an explicit function of x.

(16) 6. Determine whether each of the following improper integrals converges or diverges. Calculate the value of any convergent integral.

(a) $\displaystyle\int_0^\infty xe^{-3x^2}\,dx$

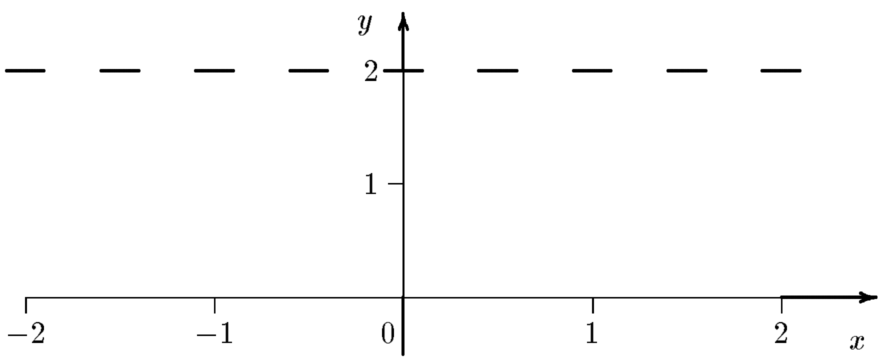
(b) $\displaystyle\int_{-2}^0 \frac{dx}{2+x}$

(10) 7. Here is part of the direction field for $\dfrac{dy}{dx} = \dfrac{y(y-2)^2(y+1)}{4}$:



DO NOT TRY TO FIND EXPLICIT SOLUTIONS OF THIS EQUATION.

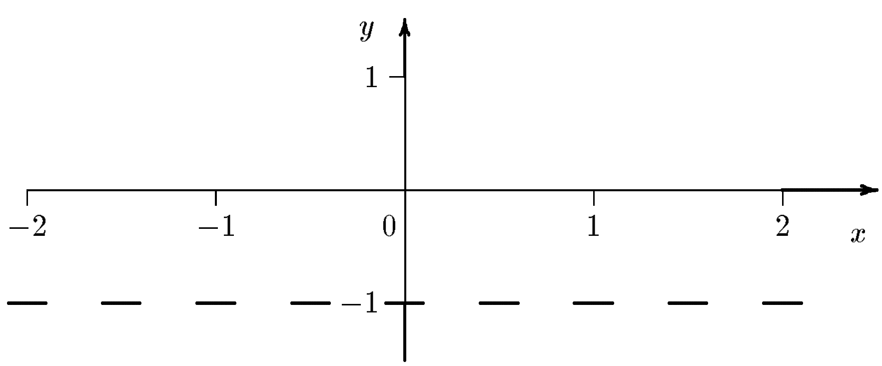
a) On the axes below sketch a solution of the equation that satisfies $0 < y(0) < 2$ and use your graph to find the limits of this solution as $x \to \pm\infty$.



$\lim_{x\to-\infty} y(x) =$ ________

$\lim_{x\to\infty} y(x) =$ ________

b) On the axes below sketch a solution of the equation that satisfies $-1 < y(0) < 0$ and use your graph to find the limits of this solution as $x \to \pm\infty$.



$\lim_{x\to-\infty} y(x) =$ ________

$\lim_{x\to\infty} y(x) =$ ________

(12) 8. Let $f(x) = \dfrac{3x}{2+5x^4}$.

(a) Find the power series representation (in powers of x) for the function $f(x)$. Write out the first three nonzero terms and give an explicit formula for the general term of this series.

(b) What is the radius of convergence of the series in (a)?

(8) 9. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \dfrac{3^n (n!)^2}{(2n)!} x^n$. Give details.

(18) 10. Let $f(x) = x^3 e^{-x^2}$. You may use the power series for the exponential function in this problem without further justification.

(a) Write out the first three nonzero terms of the power series for $f(x)$ (in powers of x) and give an explicit formula for the general term of this series.

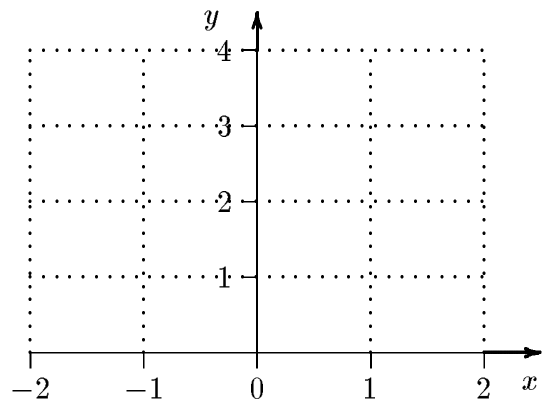
(b) Use the first three terms of the series from (a) to obtain an approximate numerical value for the definite integral $\int_0^1 x^3 e^{-x^2}\, dx$.

(c) How accurate is the approximation you obtained in (b)? Justify your answer by using an appropriate method to estimate the tail of an infinite series.

(18) 11. You are given the function $f(x) = e^{\sin x}$ and its derivatives $f'(x) = \cos(x)f(x)$, $f^{(2)}(x) = (\cos^2(x) - \sin(x))f(x)$, $f^{(3)}(x) = (\cos^3(x) - 3\cos(x)\sin(x) - \cos(x))\, f(x)$.

(a) Use the given information to find the Taylor polynomial $T_2(x)$ of degree 2 centered at $a = 0$ for the function $f(x)$.

(b) Sketch the graphs of $y = f(x)$ and $y = T_2(x)$ for $-2 \le x \le 2$ on the axes below. Label the curves.

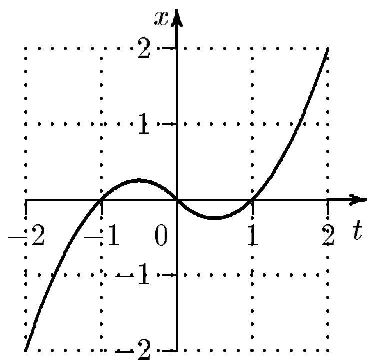


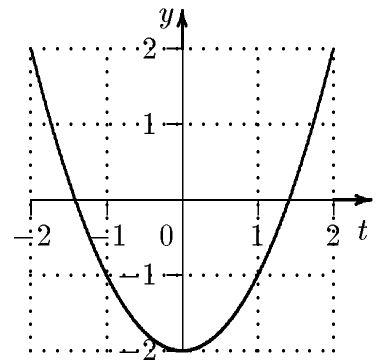
(c) Find a number M so that $|f^{(3)}(z)| \le M$ for all z in the interval $[-2, 2]$. A simple order-of-magnitude estimate for M justified by the given formulas is acceptable.

(d) Use the remainder formula in Taylor's Theorem and your value of M from (c) to find a positive number B so that $|f(x) - T_2(x)| \le 10^{-2}$ for all x in the interval $-B \le x \le B$.

(10) 12. Let $f(x) = (2 - 3x + x^2)\cos(x)$. Use the power series for $\cos x$ to calculate the coefficients c_0, c_1, c_2 and c_3 in the power series $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$.
YOU DO NOT HAVE TO CALCULATE ANY DERIVATIVES.

(14) 13. A curve is described by parametric equations $x = f(t)$, $y = g(t)$ where the graphs of x as a function of t and y as a function of t are given below:





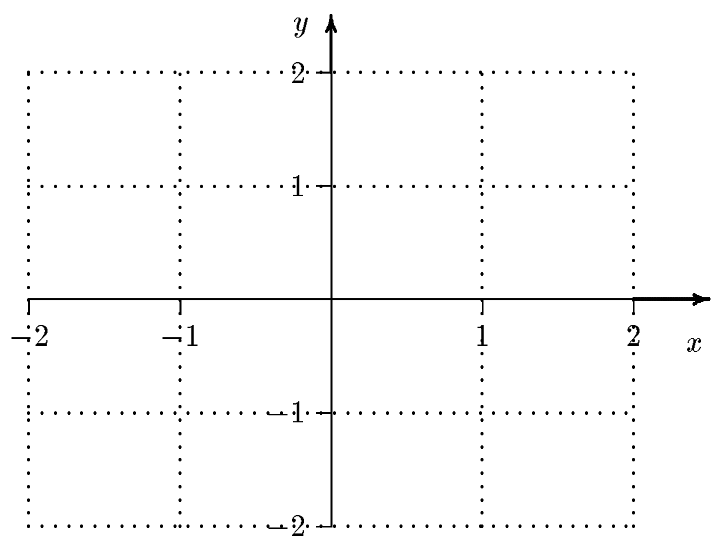
a) Fill in the following table with values estimated from the graphs.

t	-2	-1	0	1	2
x					
y					

(b) The tangent line to the curve $x = f(t)$, $y = g(t)$ is horizontal when $t =$ ____________ (give all possible values estimated from the graphs).

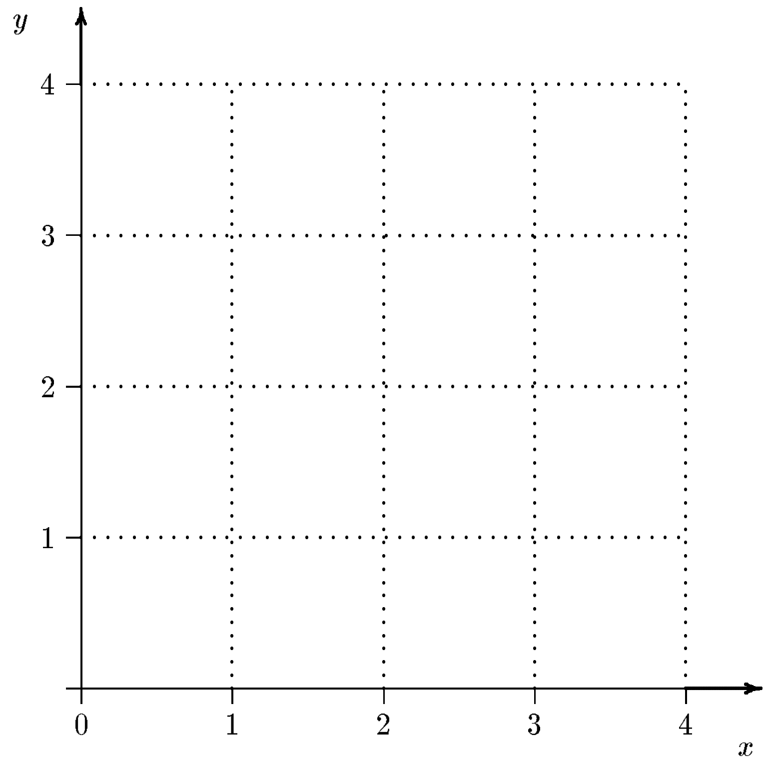
(c) The tangent line to the curve $x = f(t)$, $y = g(t)$ is vertical when $t =$ ____________ (give all possible values estimated from the graphs).

(d) Sketch the curve on the xy-plane below using all the information above.



(14) 14. Consider the curve given in polar coordinates by $r = 5\sin 2\theta$ for $0 \le \theta \le \pi/2$.

(a) Sketch the graph of this curve on the xy-plane below.



(b) Use calculus to find the area of the region bounded by this curve.