Math 152, Fall 2003
Exam 2 review answers

December 8, 2003

1. (a) Sketch the region R bounded by the curves $y = x^2 + 1$ and $y = 3 - x^2$.
   (b) Find the area of R.
   (c) Compute the volume which results when the region R is revolved about the x-axis.

   Solution: After sketching the region, you will see that the two curves intersect at $(-1, 2)$ and $(1, 2)$, so the area of R is given by integrating the difference of the two functions from -1 to 1:

$$\begin{aligned} \text{area of } R &= \int_{-1}^{1} \left(3 - x^2\right) - \left(x^2 + 1\right)\ dx, \\ &= \int_{-1}^{1} 2 - 2x^2\ dx, \\ &= 4 - \frac{4}{3}, \\ &= \frac{8}{3}. \end{aligned}$$

   For part (c), sketch the resulting region. The washer method is the most natural to use here. So, the outer radius (sometimes denoted R) will be $3 - x^2$ and the inner radius (sometimes denoted r) will be $x^2 + 1$. This gives us

$$\begin{aligned} \text{volume of revolved solid} &= \pi \int_{-1}^{1} \left(3 - x^2\right)^2 - \left(x^2 + 1\right)^2\ dx, \\ &= \pi \int_{-1}^{1} 8 - 8x^2\ dx, \\ &= \pi \left(16 - \frac{16}{3}\right), \\ &= \frac{32\pi}{3}. \end{aligned}$$

2. Integrate the following:

   (a) $\displaystyle\int \frac{15x}{(x-4)(x^2+4)}\,dx$

   (b) $\displaystyle\int \frac{7}{2x\sqrt{x^2+9}}\,dx$

   (c) $\displaystyle\int \sin^4 x \cos^5 x\,dx$

   (d) $\displaystyle\int x^2 e^{-7x}\,dx$

   (e) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

   (f) $\displaystyle\int x^2 \ln x\,dx$

Solution to (a): Start with partial fractions. Remember, we want to find A, B, and C so that

$$\frac{15x}{(x-4)(x^2+4)} = \frac{A}{x-4} + \frac{Bx+C}{x^2+4}.$$

(Why did I put $Bx+C$ above x^2+4?) Multiplying out, this becomes

$$15x = A(x^2+4) + (Bx+C)(x-4).$$

Now just solve this for A, B, and C. One way is to multiply it all out, getting

$$15x = (A+B)x^2 + (-4B+C)x + (4A-4C).$$

Now we equate coefficients on both sides to see that

$$\begin{aligned} 0 &= (A+B)x^2, \\ 15x &= (-4B+C)x, \\ 0 &= (4A-4C). \end{aligned}$$

So $B = -A$, and $C = A$ (from the first and third equalities above). The second equality then says that

$$15 = -4B + C = -4(-A) + A = 5A,$$

so $A = 3$, $B = -3$, and $C = 3$. This gives

$$\int \frac{15x}{(x-4)(x^2+4)}\,dx = \int \frac{3}{x-4}\,dx + \int \frac{-3x+3}{x^2+4}\,dx,$$

and now we just have to integrate. The first integral, $\displaystyle\int \frac{3}{x-4}\,dx$, can be done with a u-substitution; if we set $u = x-4$, $du = dx$, and thus

$$\int \frac{3}{x-4}\,dx = 3\int \frac{1}{x-4}\,dx = 3\int \frac{1}{u}\,du = 3\ln(u) = 3\ln(x-4) + \text{Constant}.$$

This leaves us with $\int \frac{-3x+3}{x^2+4}\,dx$. Break this into two integrals:

$$\int \frac{-3x+3}{x^2+4}\,dx = \int \frac{-3x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx.$$

You can do $\int \frac{-3x}{x^2+4}\,dx$ with a u-substitution, and you can do $\int \frac{3}{x^2+4}\,dx$ by remembering that

$$\int \frac{1}{x^2+1} = \arctan(x) + \text{Constant}.$$

(You'll also have to factor that 4 out of the denominator.) The final answer is:

$$\int \frac{15x}{(x-4)(x^2+4)}\,dx = 3\ln(x-4) - \frac{3}{2}\ln\left(x^2+4\right) + \frac{3}{2}\arctan\left(\frac{x}{2}\right) + \text{Constant}.$$

Solution to (b): We are supposed to find

$$\int \frac{7}{2x\sqrt{x^2+9}}\,dx.$$

You might start by trying a u-substitution, but hopefully you will find quickly that that won't work. Instead we have to use a trig substitution. I would get the constants out of the way before proceeding any further:

$$\int \frac{7}{2x\sqrt{x^2+9}}\,dx = \frac{7}{2}\int \frac{dx}{x\sqrt{x^2+9}}\,dx.$$

We substitute $x = 3\tan\theta$, so $dx = 3\sec^2\theta\;d\theta$, and we get

$$\frac{7}{2}\int \frac{\sec^2\theta\;d\theta}{\tan\theta\sqrt{9\tan^2\theta+9}}.$$

(I cancelled some 3s here.) Of course, the whole reason for this substitution is that

$$9\tan^2\theta + 9 = 9\sec^2\theta,$$

so our integral becomes

$$\begin{aligned}
\frac{7}{2}\int \frac{\sec^2\theta\;d\theta}{\tan\theta\,(3\sec\theta)} &= \frac{7}{6}\int \frac{\sec\theta\;d\theta}{\tan\theta}, \\
&= \frac{7}{6}\int \frac{\cos\theta}{\sin\theta\cos\theta}\;d\theta, \\
&= \frac{7}{6}\int \frac{1}{\sin\theta}\;d\theta.
\end{aligned}$$

Okay. Sorry for putting this on the review sheet. Or maybe I'm not. You might remember from problem #39 in Section 7.2 that

$$\int \frac{1}{\sin\theta}\, d\theta = \int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta|,$$

That was kindof a joke, so let me refresh your memory. We want to find

$$\int \frac{1}{\sin\theta}\, d\theta,$$

which is the same thing as $\displaystyle\int \csc\theta\, d\theta$.

(You might want to stop reading now and find this integral... it seems like a nice exam problem.)

To find the integral

$$\int \csc\theta\, d\theta,$$

we begin by multiplying by

$$1 = \frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}.$$

(What, you say? Well, that's the only way I know how to do it.) Now we make a u-substitution. Set $u = \csc\theta + \cot\theta$, so $du = -(\csc\theta\cot\theta + \csc^2\theta)\, d\theta$. This makes our integral

$$-\int \frac{du}{u}\, du,$$

so the integral is $-\ln|u| = -\ln|\csc\theta + \cot\theta|$.

This means that the answer to the question we really cared about, $\displaystyle\frac{7}{6}\int \frac{1}{\sin\theta}\, d\theta$, is

$$-\frac{7}{6}\ln|\csc\theta + \cot\theta| + C.$$

But of course we have to leave our answer in terms of x, not θ. To do this, you should imagine θ as an angle in a right triangle such that $x = 3\tan\theta$ (this was our original substitution). So,

$$\frac{x}{3} = \frac{\text{opposite}}{\text{adjacent}},$$

and we might as well assume that the length of the adjacent side is 1, so the length of the opposite side is $x/3$. This (by the Pythagorean Theorem) implies that the length of the hypotenuse is $\sqrt{\left(\frac{x}{3}\right)^2 + 1}$. So:

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{\sqrt{\left(\frac{x}{3}\right)^2 + 1}}{\frac{x}{3}} = \frac{\sqrt{x^2+9}}{x},$$

and

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\frac{x}{3}} = \frac{3}{x}.$$

This means that our final answer is

$$-\frac{7}{6}\ln\left|\frac{\sqrt{x^2+9}}{x} + \frac{3}{x}\right| + C.$$

Solution to (c): We are asked to find

$$\int \sin^4 x \cos^5 x \; dx.$$

Your book (page 478) gives you a plan for integrating $\int \sin^n x \cos^m x \; dx$. So, we'll follow that when integrating $\int \sin^4 x \cos^5 x \; dx$. It recommends in this case (where the power of cosine is odd) to save one factor of cosine (this will become our du), and express the rest in terms of sine by using $\cos^2 x = 1 - \sin^2 x$. This gets us:

$$\begin{aligned}
\int \sin^4 x \cos^5 x \; dx &= \int \sin^4 x \left(1 - \sin^2 x\right)^2 \cos x \; dx, \\
&= \int \sin^4 x \left(1 - 2\sin^2 x + \sin^4 x\right) \cos x \; dx, \\
&= \int \left(\sin^4 x - 2\sin^6 x + \sin^8 x\right) \cos x \; dx,
\end{aligned}$$

now we take $u = \sin x$ (so $du = \cos x \; dx$):

$$\begin{aligned}
\int \left(\sin^4 x - 2\sin^6 x + \sin^8 x\right) \cos x \; dx &= \int u^4 - 2u^6 + u^8 \; du, \\
&= \frac{u^5}{5} - 2\frac{u^7}{7} + \frac{u^9}{9} + C, \\
&= \frac{\sin^5 x}{5} - 2\frac{\sin^7 x}{7} + \frac{\sin^9 x}{9} + C.
\end{aligned}$$

(I don't know if any of you use it, but if you do, Maple does not give you the answer in this form... it gives you

$$-\frac{1}{9}\sin^3 x \cos^6 x - \frac{1}{21}\sin x \cos^6 x + \frac{1}{105}\sin x \cos^4 x + \frac{4}{315}\sin x \cos^2 x + \frac{8}{315}\sin x + C$$

instead.)

Solution to (d): To find

$$\int x^2 e^{-7x} \; dx$$

we will integrate by parts, twice, each time differentiating the x term and integrating the e term. First we use

$$u = x^2, \qquad v = \frac{1}{-7}e^{-7x},$$
$$du = 2x\ dx, \qquad dv = e^{-7x}\ dx.$$

This gives us

$$\begin{aligned}\int x^2 e^{-7x}\ dx &= -\frac{1}{7}x^2e^{-7x} - \int 2x\frac{1}{-7}e^{-7x}\ dx,\\ &= -\frac{1}{7}x^2e^{-7x} + \frac{2}{7}\int xe^{-7x}\ dx.\end{aligned}$$

Now we can just use integration by parts again to find $\int xe^{-7x}\ dx$. The final answer is

$$\int x^2e^{-7x}\ dx = -\frac{1}{7}x^2e^{-7x} - \frac{2}{49}xe^{-7x} - \frac{2}{343}e^{-7x} + C.$$

Solution to (e): This one, $\int \frac{(\ln x)^2}{x}\ dx$, is the easiest of the integrals. We can do it with a simple u-substitution: $u = \ln x$ so $du = \frac{1}{x}\ dx$. This gives us

$$\int \frac{(\ln x)^2}{x}\ dx = \int (\ln x)^2\frac{1}{x}\ dx = \int u^2\ du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.$$

Solution to (f): We are asked to find

$$\int x^2 \ln x\ dx.$$

We can do this with integration by parts, taking

$$u = \ln x, \qquad v = \frac{x^3}{3},$$
$$du = \frac{1}{x}\ dx, \qquad dv = x^2\ dx.$$

Then we have

$$\begin{aligned}\int x^2\ln x\ dx &= \frac{x^3\ln x}{3} - \int \frac{x^3}{3}\frac{1}{x}\ dx,\\ &= \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2\ dx,\\ &= \frac{x^3\ln x}{3} - \frac{1}{3}\frac{x^3}{3} + C,\\ &= \frac{x^3\ln x}{3} - \frac{x^3}{9} + C.\end{aligned}$$

3. Solve the initial value problem

$$\frac{dy}{dx} = \frac{-3\sin 3x}{y}, \qquad y(0) = 1$$

explicity as a function of x.

NOTE: There is a typo on the review sheet I gave you. In that version the left-hand side of the differential equation was dx/dy. Although you can solve that differential equation (you'll get $y = C\,(\csc 3x - \cot 3x)^{-1/9}$), it is harder than it was meant to be, and there is no way to satisfy the initial conditions (in that case y isn't even defined at 0). The problem is correctly stated above.

Solution: We begin by separating the variables, putting the ys on one side and the xs on the other:

$$\frac{dy}{y} = -3\sin 3x\; dx.$$

Now we integrate both sides. The integral of the left-hand side is $\ln y$. The integral of the right-hand side is $\cos 3x + C$, so

$$\ln y = \cos 3x + C,$$

and thus

$$y = e^{\cos 3x + C} = e^C e^{\cos 3x} = De^{\cos 3x}.$$

(Here I replaced e^C by D; they are both just unknown constants at this point.) This is the general solution, but we were asked for a solution to the initial value problem, so we have to find D. We do that by substituting 0 in:

$$y(0) = De^{\cos 3\cdot 0} = De^1,$$

since we are supposed to have $y(0) = 1$, we get $De = 1$, so $D = 1/e$. The final solution is then

$$y(x) = \frac{1}{e}e^{\cos 3x}.$$

4. Find the length of the curve given by

$$x = e^{2t} - 2t, \qquad y = 4e^t$$

between $1 \leq t \leq 3$.

Solution: Remember that we know two formulas for arc length: one for when the curve is in the form $y = f(x)$, in which case the arc length is given by

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx,$$

and one for when the curve is parametric (when both x and y depend on some other variable, often t), and in this case the formula is

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt,$$

In this problem we have a parametric curve, so we use that formula. We have

$$\begin{aligned} \frac{dx}{dt} &= 2e^{2t} - 2, \\ \frac{dy}{dt} &= 4e^t. \end{aligned}$$

This makes the arc length integral

$$\begin{aligned} \int_1^3 \sqrt{(2e^{2t}-2)^2 + (4e^t)^2}\ dt, &= \int_1^3 \sqrt{4e^{4t} - 8e^{2t} + 4 + 16e^{2t}}\ dt, \\ &= \int_1^3 \sqrt{4e^{4t} + 8e^{2t} + 4}\ dt, \\ &= \int_1^3 \sqrt{(2e^{2t}+2)^2}\ dt, \\ &= \int_1^3 2e^{2t} + 2\ dt, \\ &= e^6 - e^2 + 4. \end{aligned}$$

5. (a) Sketch the two curves $r = 1 + 2\sin\theta$ and $r = 4\sin\theta$, $0 \le \theta \le \pi$, labelling the x and y-coordinated of the points of intersection.

   (b) Set up but DO NOT EVALUATE an integral representing the area of the region outside $r = 1 + 2\sin\theta$ but inside $r = 4\sin\theta$.

   NOTE: There was a typo in the statement of (b) in the copy I gave you. The corrected version appears above.

   Solution: Coming soon.

6. For each of the following series, tell whether they are absolutely convergent, conditionally convergent, or divergent.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+3}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^2}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^{1/n}}$

(d) $\sum_{n=34}^{\infty} \frac{1+e^n}{7^n}$

(e) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Solution to (a): $\sum_{n=0}^{\infty} \frac{\sqrt{n}}{n^2+3}$ is convergent by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

Solution to (b): You could note first that $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^2}$ is convergent by the Alternating Series Test. It is also absolutely convergent, because the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3n^2} \right| = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent (this is a p-series, so you should know that it converges, but you should also be able to prove this fact using the Integral Test).

Solution to (c): For $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n^{1/n}}$ to converge, its terms must tend to 0 (this is called the Test for Divergence). Do they?

Let's start by figuring out what $n^{1/n}$ does. Write it as

$$n^{1/n} = e^{\ln(n^{1/n})}.$$

This might seem like a silly manipulation right now, but using the rules of logarithms we can rewrite this as

$$e^{\frac{1}{n}\ln(n)} = e^{\frac{\ln n}{n}}.$$

Using L'Hopital's rule,

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^0 = 1.$$

From this it follows that the limit as n goes to infinity of $\frac{(-1)^n}{3n^{1/n}}$ does not exist, so the series does not converge.

Solution to (d): $\sum_{n=34}^{\infty} \frac{1+e^n}{7^n}$ converges by limit comparison with $\sum_{n=34}^{\infty} \frac{e^n}{7^n} = \sum_{n=34}^{\infty} \left(\frac{e}{7}\right)^n$, which is a geometric series with ratio $e/7 < 1$.

Solution to (e): It is not hard to show that $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges, but it is slightly subtle. Do NOT compare it with

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

right off the bat, because one of the conditions needed to apply the Comparison Test is that the terms of your series are positive, and $\frac{\sin n}{n^2}$ is negative sometimes.

(Consider this: $\sum_{n=1}^{\infty} -1$ does not converge, yet each term is less than 0 and $\sum_{n=1}^{\infty} 0 = 0$ clearly converges.)

Instead we show that the series converges absolutely. It converges absolutely because

$$\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

7. How many terms of the series

$$\sum_{n=1}^{\infty} \frac{4}{n^{5/2}}$$

are required to approximate the sum of the series to within 0.001?

Solution: The question asks us to find some number N so that

$$\sum_{n=1}^{\infty} \frac{4}{n^{5/2}} - \sum_{n=1}^{N} \frac{4}{n^{5/2}} \leq 0.001.$$

We can simplify this inequality to get

$$\sum_{n=N+1}^{\infty} \frac{4}{n^{5/2}} \leq 0.001.$$

(Make sure you understand the cancellation there.)

To get a bound on this sum from $N+1$ to infinity we compare the sum to an integral:

$$\sum_{n=N+1}^{\infty} \frac{4}{n^{5/2}} \leq \int_{N}^{\infty} \frac{4}{x^{5/2}}\, dx,$$

and then

$$\begin{aligned}\int_N^\infty \frac{4}{x^{5/2}}\,dx &= \lim_{t\to\infty} 4\int_N^t x^{-5/2}\,dx,\\ &= 4\lim_{t\to\infty} -\frac{2}{3}t^{-3/2}+\frac{2}{3}N^{-3/2},\\ &= \frac{8}{3N^{3/2}}.\end{aligned}$$

So now we just have to make $8/3N^{3/2}$ less than 0.001. This occurs if N is at least 193. It would also be fine if you left your answer as $N \geq \left(\frac{8}{3\cdot 0.001}\right)^{2/3}$.

Bonus (easier) question: What about the same question, but with $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{5/2}}$ as the series? The answer is that you need N to be at least 16.

8. Find the interval of convergence for each of the following series

   (a) $\sum_{n=1}^{\infty}\frac{(3x)^n}{n+4}$

   (b) $\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}x^n$

   (c) $\sum_{n=0}^{\infty}\frac{n^2x^n}{(3n+1)!}$

   (d) $\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{n2^n}$

Solution to (a): To find the interval of convergence for $\sum_{n=1}^{\infty}\frac{(3x)^n}{n+4}$ we just use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{n+1\text{st term}}{n\text{th term}}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(3x)^{n+1}}{n+5}}{\frac{(3x)^n}{n+4}}\right|,\\ &= \lim_{n\to\infty}\left|\frac{n+4}{n+5}3x\right|,\\ &= |3x|\lim_{n\to\infty}\frac{n+4}{n+5} = |3x|,\end{aligned}$$

so the series converges when $|x| < \dfrac{1}{3}$. Remember that we also have to test the endpoints. At $x = 1/3$, we get the series

$$\sum_{n=1}^{\infty} \frac{1}{n+4},$$

which diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$. At $x = -1/3$ we get the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4},$$

which converges by the Alternating Series Test. So, the interval of convergence is $[-1/3, 1/3)$, and the radius of convergence is $R = 1/3$.

Solution to (b): We use the Ratio Test again on $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$, and we get

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(2n+2)!} x^{n+1}}{\frac{(n!)^2}{(2n)!} x^n} \right| &= |x| \lim_{n\to\infty} \frac{((n+1)!)^2}{(n!)^2} \cdot \frac{(2n)!}{(2n+2)!}, \\
&= |x| \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}, \\
&= |x| \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2}, \\
&= \frac{|x|}{4},
\end{aligned}$$

so the series converges when $|x| < 4$.

I'll trust you to check the endpoints; the series doesn't converge at either of them, so the interval of convergence is $(-4, 4)$ and the radius of convergence is $R = 4$.

Solution to (c): The series is $\sum_{n=0}^{\infty} \frac{n^2 x^n}{(3n+1)!}$, and we use the Ratio Test once again:

$$|x| \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{(3n+1)!}{(3(n+1)+1)!} = |x| \lim_{n\to\infty} \frac{(n+1)^2}{n^2(3n+4)(3n+3)(3n+2)} = 0$$

(for all x), so the series converges for all x. This makes its interval of convergence $(-\infty, \infty)$ and its radius of convergence $R = \infty$.

Solution to (d): Here we have $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n 2^n}$, and we use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(n+1)2^{n+1}} \cdot \frac{n 2^n}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \left| -\frac{x^2}{2} \cdot \frac{n}{n+1} \right| = \frac{x^2}{2},$$

so the series converges when

$$\frac{x^2}{2} < 1,$$

that is, when $-\sqrt{2} < x < \sqrt{2}$.

Now we test the endpoints. At $x = \sqrt{2}$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}^{2n}}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \left(\sqrt{2}^2\right)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test.

At $x = -\sqrt{2}$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n \left(-\sqrt{2}\right)^{2n}}{n2^n},$$

which works out to the same thing, so the series converges here as well. So the interval of convergence is $[-\sqrt{2}, \sqrt{2}]$, and the radius of convergence is $R = \sqrt{2}$.

9. Use the Maclaurin series for e^x to get an infinite series for the integral

$$I = \int_0^1 e^{-x^2/2}\, dx.$$

Solution: The Maclaurin series (remember that a Maclaurin series is just a Taylor series centered at 0) for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(You should be able to show that this converges for all x.)

To get the Maclaurin series for $e^{-x^2/2}$, we just plug $-x^2/2$ into the Maclaurin series for e^x. This gives us

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-x^2/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}.$$

Now we just integrate this term by term:

$$\begin{aligned}
\int_0^1 e^{-x^2/2}\, dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}\, dx, \\
&= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n x^{2n}}{2^n n!}\, dx, \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}.
\end{aligned}$$

IMPORTANT: Things I forgot to put in the review sheet that you should review:

1. Improper integrals.
2. The Remainder Theorem for Taylor series.
3. The Remainder Theorem for Alternating Series.