Numbers (*Rough estimates.*)

$\pi \approx 3 \quad e \approx 2.7 \quad \sqrt{2} \approx 1.4 \quad \sqrt{3} \approx 1.7$

$\ln 2 \approx .7 \quad \ln 10 \approx 2.3$

$n! \approx (n/e)^n$ (roughly)

Trigonometry

$\tan^2 x + 1 = \sec^2 x$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\sin(\pi/3) = \sqrt{3}/2$

$\sin(\pi/4) = \sqrt{2}/2$

$\sin(\pi/6) = \sqrt{1}/2$

Differentiation

$f(u)' = f'(u) \cdot u'$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \csc x = -\csc x \cot x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\csc^2 x$

$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$

$a^x = e^{x \ln a}$

Also used for **antidifferentiation**.

Parametric: $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$

Integration

$\int u dv = uv - \int v du$

$\int \sec x \; dx = \ln x|\sec x + \tan x| + C$

$\int \csc x \; dx = -\ln x|\csc x + \cot x| + C$

$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C$

Forms: $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$

Trig substitutions involving sin, tan, sec.

Numerical Integration

Midpoint rule: sample function at midpoints.

Trapezoidal rule

(average left and right endpoint rules):

coefficients $[1, 2, 2, \ldots, 2, 2, 1]$; and $/2$.

Simpson's rule

(interpolate parabolic segments, n even):

coefficients $[1, 4, 2, 4, \ldots, 4, 2, 4, 1]$; and $/3$.

Error estimates:

$E_T \leq \frac{K_2(b-a)^3}{12n^2}$

$E_M \leq \frac{K_2(b-a)^3}{24n^2}$

$E_S \leq \frac{K_4(b-a)^5}{180n^4}$

Geometry

Area: $A = \int dA$

$dA = hdx$ (vertical) or wdy (horizontal)

or $\frac{1}{2}r^2 d\theta$ (polar)

Volume: $V = \int dV$

$dV = Adx$ (vertical) or Ady (horizontal)

or $dV = 2\pi rhdx$ (cylindrical shells)

Solids of revolution:

$A = \pi r^2$ (disc) or $\pi r^2_{\text{out}} - \pi r^2_{\text{in}}$ (annulus)

Arc length: $s = \int ds$

$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}dx$

Surface area: $S = \int dS$

Surface of revolution: $dS = 2\pi r ds$

Differential equations

$y' = dy/dx$ (Leibniz notation)

Separate if possible

Use initial conditions, if known

$y' = ky$ corresponds to $y = Ce^{kt}$.

Applications:

Compound interest

Population, Radioactivity

Cooling (Newton's Law)

Orthogonal trajectories:

replace the slope y' by $-1/y'$.

Polar coordinates

$r = \sqrt{x^2 + y^2}$ $\quad \theta = \arctan y/x$

$x = r\cos\theta \quad y = r\sin\theta$

$dy/dx = \dfrac{dy/d\theta}{dx/d\theta}$

Sequences

Some limits:

$\lim_{n\to\infty} n^{1/n} = 1$

$\lim_{n\to\infty} (1 + 1/n)^n = e$

Series: convergence, and values

Geometric:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \text{ for } |x| < 1.$$

Necessary for convergence: $\lim_{n\to\infty} a_n = 0$.

Tests applied for *absolute convergence*:

1. Standard series $a_n = 1/n^p$, $p > 1$; or geometric
2. Comparison and limit comparison.
3. Ratio
4. Root
5. Integral
6. Telescoping (rare)

Test applied for *conditional convergence*:

Alternating, $|a_n|$ decreases monotonically to 0.

Series: error estimates

When the alternating series test applies:

$|s - s_n| \le |a_{n+1}|$.

Taylor series

$$T_a f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x)}{i!}(x-a)^i$$

Standard examples:

e^x, $\sin x$, $\cos x$: from Taylor's rule; x arbitrary

$\arctan x$, $\ln(1+x)$

gotten by integrating geometric series; $|x| < 1$

Expand $\dfrac{1}{1+x^2}$ *or* $\dfrac{1}{1+x}$

as a geometric series; then integrate.

Newton's Binomial Formula:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n.$$

$$\binom{k}{n} = \frac{k(k-1)\cdot \ldots \cdot (k-(n-1))}{n!}$$

Interval of convergence:

absolute convergence within the interval,

divergence outside, and

unclear behavior at endpoints.