MAXIMAL INDEPENDENT SETS AND SEPARATING COVERS

VINCENT VATTER
Department of Mathematics
University of Florida
Gainesville, Florida USA

In 1973, Katona raised the problem of determining the maximum number of subsets in a separating cover on n elements. The answer to Katona's question turns out to be the inverse to the answer to a much simpler question: what is the largest integer which is the product of positive integers with sum n? We give a combinatorial explanation for this relationship, via Moon and Moser's answer to a question of Erdős: how many maximal independent sets can a graph on n vertices have? We conclude by showing how Moon and Moser's solution also sheds light on a problem of Mahler and Popken's about the complexity of integers.

1. INTRODUCTION

We begin with a simply stated problem, which has made numerous appearances in mathematics competitions:[1] what is the largest number which can be written as the product of positive integers that sum to n?

We denote this number by $\ell(n)$. A moment's thought shows that one should use as many 3s as possible; if $m \geq 5$ appears in the product then it can be replaced by $3(m-3) > m$, and while 2s and 4s can occur in the product, the latter can occur at most once since $4 \cdot 4 < 2 \cdot 3 \cdot 3$ and the former at most twice since $2 \cdot 2 \cdot 2 < 3 \cdot 3$. This shows that for $n \geq 2$,

$$\ell(n) = \begin{cases} 3^i & \text{if } n = 3i, \\ 4 \cdot 3^{i-1} & \text{if } n = 3i+1, \\ 2 \cdot 3^i & \text{if } n = 3i+2, \end{cases}$$

Date: August 26, 2010
Key words and phrases. integer complexity, maximal independent set, separating covers
AMS 2000 Subject Classification. 05C35, 05C69, 05D05, 11A99

[1] In particular, the 1976 IMO asked for the $n = 1976$ case, the 1979 Putnam asked for the $n = 1979$ case, and on April 23rd 2002, the 3rd Community College of Philadelphia Colonial Mathematics Challenge asked for the $n = 2002$ case.

while $\ell(1) = 1$. Note that it follows from the combinatorial definition of ℓ that this function is strictly increasing and *super-multiplicative*, meaning that it satisfies $\ell(n_1)\ell(n_2) \leq \ell(n_1 + n_2)$.

In 1973, G. O. H. Katona [6, Problem 8, p. 306] posed a problem which looks completely unlike the determination of $\ell(n)$. A *separating cover*[2] over the ground set X is a collection $\mathcal{S}$ of subsets of X which satisfies two properties:

- the union of the sets in $\mathcal{S}$ is all of X, and
- for every pair of distinct elements $x, y \in X$ there are disjoint sets $S, T \in \mathcal{S}$ with $x \in S$ and $y \in T$.

Katona asked about the function

$$s(m) = \min\{n : \text{there is a separating cover on } m \text{ elements with } n \text{ sets}\}.$$

M.-C. Cai and A. C. C. Yao gave independent solutions several years later.

Theorem 1 (Cai [2] and Yao [9], independently)**.** *For all $m \geq 2$,*

$$s(m) = \begin{cases} 3i & \text{if } 2 \cdot 3^{i-1} < m \leq 3^i, \\ 3i+1 & \text{if } 3^i < m \leq 4 \cdot 3^{i-1}, \\ 3i+2 & \text{if } 4 \cdot 3^{i-1} < m \leq 2 \cdot 3^i, \end{cases}$$

while $s(1) = 1$.

Thus $s(\ell(n)) = n$ for all positive integers n — in other words, s is a left inverse of ℓ. Ironically, the question we began with appears at the beginning of R. Honsberger's *Mathematical Gems III* [5], while Katona's problem occurs at the end, where Honsberger describes the proof as "long and much more complicated than the arguments in the earlier chapters." We present a short combinatorial explanation for the equivalence of these two problems.

2. A COMBINATORIAL INTERPRETATION OF ℓ

In order to give a combinatorial explanation for why $s(\ell(n)) = n$, we first need a combinatorial interpretation of ℓ. We use a graph-theoretic interpretation, although several others are available.[3] Let G be a graph over the vertex set $V(G)$. A subset $I \subseteq V(G)$ is *independent* if there is no edge between any two vertices of I, and it is a *maximal independent set (MIS)* if it is not properly contained in any other independent set. In the 1960s, P. Erdős asked how many MISes a graph on n vertices could have, which we define as

$$g(n) = \max\{m : \text{there is graph on } n \text{ vertices with } m \text{ MISes}\}.$$

[2] We make this slight deviation from Katona's original formulation so that $s(1) = 1$.

[3] Another — in terms of integer complexity — is given later in this note. Additionally, $\ell(n)$ is the order of the largest abelian subgroup of the symmetric group of order n; see Bercov and Moser [1].

Let us denote by $m(G)$ the number of MISes in the graph G. This quantity is particularly easy to compute when G is a disjoint union:

Proposition 2. *The disjoint union of the graphs G and H has $m(G)m(H)$ MISes.*

Proof. For any MIS M of this union, $M \cap V(G)$ must be an MIS of G and $M \cap V(H)$ must be an MIS of H. Conversely, if M_G and M_H are MISes of G and H, respectively, then $M_G \cup M_H$ is an MIS of the disjoint union of G and H. □

Because the complete graph on n vertices has n MISes, Proposition 2 implies that $g(n) \geq \ell(n)$ for all positive integers n; we need only take a disjoint union of edges, triangles, and complete graphs on 4 vertices to achieve this lower bound. In 1965, J. W. Moon and L. Moser proved that this is best possible.

Theorem 3 (Moon and Moser [8])**.** *For all positive integers n, $g(n) = \ell(n)$.*

Indeed, Moon and Moser showed that the only extremal graphs (the graphs with $g(n)$ MISes) are those built by taking disjoint copies of edges, triangles, and complete graphs on 4 vertices in the quantities suggested by the formula for ℓ. (In the case $n = 3i + 1 \geq 4$ there are two extremal graphs, one with $i - 1$ triangles and two disjoint edges, the other with $i - 1$ triangles and a complete graph on 4 vertices.)

3. A SHORT PROOF OF THEOREM 3

Before demonstrating the relationship between MISes and separating covers, we pause to present a short proof of Moon and Moser's theorem. First we need a definition: for a set $X \subseteq V(G)$, we denote by $G - X$ the graph obtained by removing the vertices X from G and all edges incident to vertices in X. When $X = \{v\}$, we abbreviate this notation to $G - v$. Our proof makes extensive use of the following upper bound.

Proposition 4. *For any graph G and vertex $v \in V(G)$, we have*

$$m(G) \leq m(G - v) + m(G - N[v]),$$

where $N[v]$ denotes the closed neighborhood of v, i.e., v together with its neighbors.

Proof. The map $M \mapsto M - v$ gives a bijection between MISes of G containing v and MISes of $G - N[v]$. The proof is completed by noting that every MIS of G that does not contain v is also an MIS of $G - v$. □

Proof of Theorem 3. Our proof is by induction on n, and we prove the stronger statement which characterizes the extremal graphs. It is easy to check the theorem for graphs with five or fewer vertices, so take G to be a graph on $n \geq 6$ vertices, and assume the theorem holds for graphs with fewer than n vertices.

If G contains a vertex of degree 0, that is, an isolated vertex, then clearly $m(G) \leq g(n-1) = \ell(n-1) < \ell(n)$. If G contains a vertex v of degree 1 then, letting w denote the sole vertex adjacent to v, we have by Proposition 4 that

$$m(G) \leq m(G-w) + m(G-N[w]) \leq 2\ell(n-2) = \begin{cases} 8 \cdot 3^{i-2} & \text{if } n = 3i, \\ 4 \cdot 3^{i-1} & \text{if } n = 3i+1, \\ 2 \cdot 3^{i} & \text{if } n = 3i+2. \end{cases}$$

In all three cases we have an upper bound of at most $\ell(n)$, with equality if and only if $n = 3i+1$ and G is a disjoint union of $i-1$ triangles and two edges, or $n = 3i+2$ and G is a disjoint union of i triangles and an edge.

If G contains a vertex v of degree 3 or greater, then we have

$$m(G) \leq m(G-v) + m(G-N[v]) \leq \ell(n-1) + \ell(n-4) = \begin{cases} 8 \cdot 3^{i-2} & \text{if } n = 3i, \\ 4 \cdot 3^{i-1} & \text{if } n = 3i+1, \\ 16 \cdot 3^{i-2} & \text{if } n = 3i+2. \end{cases}$$

Again, all three cases give an upper bound of at most $\ell(n)$, with equality if and only if $n = 3i+1$ and G is a disjoint union of $i-1$ triangles together with a complete graph on 4 vertices.

This leaves us to consider the case where every vertex of G has degree 2, which implies that G consists of a disjoint union of cycles. If each of these cycles is a triangle, then $n = 3i$ and G is a disjoint union of i triangles, as desired. Thus we may assume that at least one connected component of G is a cycle of length $j \geq 4$, which we denote by C_j. Our goal in this case is to show that G is not extremal (i.e., $m(G) < \ell(n)$), and by the super-multiplicativity of ℓ, it suffices to show that this single cycle of length j is not extremal. It is easy to check that $m(C_4) = 2 < 4 = \ell(4)$ and $m(C_5) = 5 < 6 = \ell(5)$, it therefore suffices to show that $m(C_j) < \ell(j)$ for $j \geq 6$. (In fact, Füredi [3] found $m(C_j)$ exactly — it is the jth Perrin number.) Label the vertices of our cycle on $j \geq 6$ vertices as $u, v, w, \ldots$ so that u is adjacent to v which is in turn adjacent to w. By applying Proposition 4 twice, we see that for $j \geq 6$,

$$\begin{aligned} m(C_j) &\leq m(C_j - w) + m(C_j - N[w]) \\ &\leq m(C_j - w - u) + m(C_j - w - N[u]) + m(C_j - N[w]) \\ &\leq 2\ell(j-3) + \ell(j-4), \end{aligned}$$

which is strictly less that $3\ell(j-3) = \ell(j)$, completing the proof. □

4. A COMBINATORIAL EXPLANATION FOR $s(\ell(n)) = n$

With Moon and Moser's Theorem 3 proved, we are now ready to explain the connection to separating covers. Propositions 5 and 6 illuminate the connection between separating covers and MISes, and then Proposition 7 gives a combinatorial explanation for why s is a left inverse of $\ell = g$.

Proposition 5. *From a graph on n vertices with m MISes one can construct a separating cover on m elements with at most n sets.*

Proof. Take G to be a graph with n vertices and m MISes and let $\mathcal{M}$ denote the collection of MISes in G. The separating cover promised consists of the family of sets $\{S_v : v \in V(G)\}$ where

$$S_v = \{M \in \mathcal{M} : v \in M\}.$$

Clearly this is a family with m elements (the MISes $\mathcal{M}$) and n (not necessarily distinct) sets (one for each vertex of G), and this family covers the set $\mathcal{M}$ because each MIS lies in at least one S_v, so it remains to check only that it is separating. Take distinct sets $M, N \in \mathcal{M}$. Because M and N are both maximal there is some vertex $u \in M \setminus N$. By the maximality of N, it must contain a vertex v adjacent to u. Therefore $M \in S_u$, $N \in S_v$, and because u and v are adjacent, $S_u \cap S_v = \emptyset$, completing the proof. □

Proposition 6. *From a separating cover on m elements with n sets one can construct a graph on n vertices with at least m MISes.*

Proof. Let $\mathcal{S}$ be such a cover over the ground set X. We define a graph G on the vertices $\mathcal{S}$ where $S \in \mathcal{S}$ is adjacent to $T \in \mathcal{S}$ if and only if they are disjoint. For each $x \in X$, the set

$$I_x = \{S \in \mathcal{S} : x \in S\}$$

is an independent set in G. For each $x \in X$, choose an MIS $M_x \supseteq I_x$. We have only to show that these MISes are distinct. Take distinct elements $x, y \in X$. Because $\mathcal{S}$ is separating, there are disjoint sets $S, T \in \mathcal{S}$ with $x \in S$ and $y \in T$. Therefore $S \in M_x$, $T \in M_y$, and since S and T are disjoint they are adjacent in G, so $T \notin M_x$, and thus $M_x \neq M_y$. □

Proposition 7. *For all positive integers m and n,*

$$\begin{aligned} s(m) &= \min\{n : g(n) \geq m\}, \\ g(n) &= \max\{m : s(m) \leq n\}. \end{aligned}$$

Proof. First observe that s and g are both nondecreasing. The proof then follows from the two claims

(1) If $s(m) \leq n$ then $g(n) \geq m$, and

(2) If $g(n) \geq m$ then $s(m) \leq n$

To prove (1), suppose that $s(m) \leq n$. Then there is a separating cover with m elements and at most n sets, so by Proposition 6, there is a graph with at most n vertices and at least m MISes. This and the fact that g is nondecreasing establish that $g(n) \geq m$.

Now suppose that $g(n) \geq m$. Then there is a graph with n vertices and at least m MISes, so by Proposition 5, there is a separating cover with at least m elements and at most n sets. Because s is nondecreasing, we conclude that $s(m) \leq n$, proving (2). □

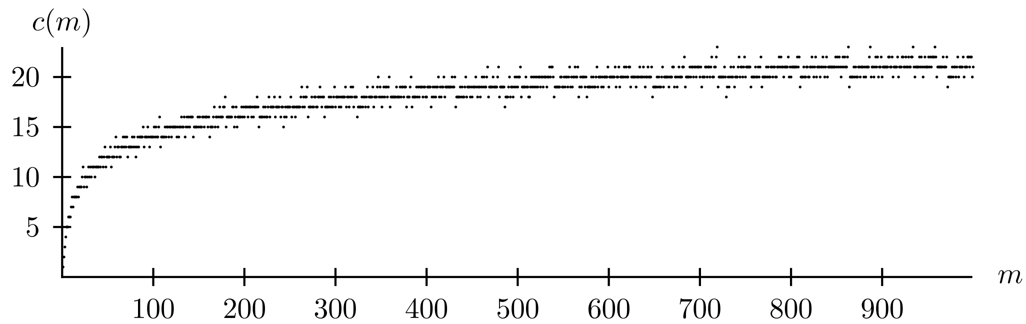

Figure 1: The complexities of the first 1000 integers.

5. INTEGER COMPLEXITY

We conclude with another appearance of g. The *complexity*, $c(m)$, of the integer m is the least number of 1s needed to represent it using only $+$s, $\cdot$s, and parentheses. For example, the complexity of 10 is 7, and there are essentially three different minimal expressions:

$$10 = (1+1+1)(1+1+1)+1 = (1+1)(1+1+1+1+1) = (1+1)((1+1)(1+1)+1),$$

Figure 1 shows a plot of the complexities of the first 1000 integers.

This definition was first considered by Mahler and Popken [7], and while a straightforward recurrence,

$$c(m) = \min\{c(d) + c(m/d) : dm\} \cup \{c(i) + c(m-i) : 1 \le i \le m-1\},$$

is easy to verify, several outstanding conjectures and questions remain, for which we refer to R. K. Guy [4]. In that article, Guy mentions that J. Selfridge gave an inductive proof of the following result.

Proposition 8 (Selfridge [unpublished])**.** *The greatest integer of complexity n is $g(n)$.*

One direction of Selfridge's proposition is clear: the problem we began with shows that $\ell(n) = g(n)$ has complexity at most n. In a final demonstration of the surprising versatility of Moon and Moser's Theorem 3, we show how it implies the other direction, via the following construction.

Proposition 9. *From an expression of the integer m with n 1s one can construct a graph on n vertices with m MISes.*

Proof. Before describing our inductive construction we need a definition. Given graphs G and H, their *join* is the graph $G + H$ obtained from their disjoint union $G \cup H$ by adding all edges connecting vertices of G with vertices of H. We know already from Proposition 2 that $m(G \cup H) = m(G)m(H)$, and a similar formula for joins is easy to verify: $m(G + H) = m(G) + m(H)$ because every MIS in $G + H$ is either an MIS of G or an MIS of H.

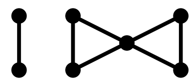

Figure 2: The construction described in the proof of Proposition 8, applied to the expression

$$10 = (1+1)((1+1)(1+1)+1).$$

There are graphs on 7 vertices with more MISes than the graph shown because 10 is not the greatest integer of complexity 7 (12 is).

Now suppose we have an expression of the integer m with n 1s. If $n = 1$, then there is only one such expression, 1, and we associate to this expression the one vertex graph. If $n \geq 2$, then any such expression must decompose as either $e_1 + e_2$ or $e_1 e_2$, where e_1 and e_2 are expressions with fewer 1s. If our expression is $e_1 + e_2$ then we associate it to the join of the graphs associated to e_1 and e_2, and if our expression is $e_1 e_2$ then we associate it to the disjoint union of the graphs associated to e_1 and e_2. Figure 2 shows an example. It follows that the resulting graph has precisely as many vertices as the expression has 1s, and precisely m MISes. □

Acknowledgements. I am grateful to the referees for their detailed and insightful comments. In particular, the change from "separating families" to "separating covers," which simplified many of these results, was suggested by one of the referees.

REFERENCES

[1] BERCOV, R., AND MOSER, L. On Abelian permutation groups. *Canad. Math. Bull. 8* (1965), 627–630.

[2] CAI, M. C. Solutions to Edmonds' and Katona's problems on families of separating subsets. *Discrete Math. 47*, 1 (1983), 13–21.

[3] FÜREDI, Z. The number of maximal independent sets in connected graphs. *J. Graph Theory 11*, 4 (1987), 463–470.

[4] GUY, R. K. Unsolved Problems: Some Suspiciously Simple Sequences. *Amer. Math. Monthly 93*, 3 (1986), 186–190.

[5] HONSBERGER, R. *Mathematical gems. III*, vol. 9 of *The Dolciani Mathematical Expositions*. Mathematical Association of America, Washington, DC, 1985.

[6] KATONA, G. O. H. Combinatorial search problems. In *A Survey of Combinatorial Theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1970)*. North-Holland, Amsterdam, 1973, pp. 285–308.

[7] MAHLER, K., AND POPKEN, J. On a maximum problem in arithmetic. *Nieuw Arch. Wiskunde (3) 1* (1953), 1–15.

[8] MOON, J. W., AND MOSER, L. On cliques in graphs. *Israel J. Math. 3* (1965), 23–28.

[9] YAO, A. C. C. On a problem of Katona on minimal separating systems. *Discrete Math. 15*, 2 (1976), 193–199.