SMALL CONFIGURATIONS IN SIMPLE PERMUTATIONS

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We prove that every entry in a simple permutation of length at least 4 is contained in a copy of 2413 or 3412 or plays the role of the '3' in a copy of 25314 or 41352. This allows us to give a short proof that the number of permutations with at most r copies of 132 has an algebraic generating function for any integer r.

1. INTRODUCTION

An *interval* in the permutation π is a set of contiguous indices $I = [a, b]$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ also forms an interval of natural numbers. Every permutation π of $[n] = \{1, 2, \ldots, n\}$ has intervals of size 0, 1, and n; π is said to be *simple* if it has no other intervals. Simple permutations have been the focus of significant study, both for their own right and as a tool in the study of permutation classes[1]; the reader is referred to Brignall's survey [3] for many references to earlier work. Recently, Brignall, Huczynska, and Vatter [4] proved a Ramsey-type result for simple permutations: every long simple permutation contains a large simple subpermutation of one of three forms. The main result of this note is much more local:

Proposition 1. *Every entry in a simple permutation π of length at least 4 is contained in a copy of 2413 or 3142 or participates as the '3' in a copy of 25314 or 41352.*

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[1]The permutation π of length n is said to *contain* the permutation σ of length k, written $\sigma \leqslant \pi$, if π has a subsequence of length k in the same relative order as σ, and such a subsequence is called a *copy* of σ. For example, $\pi = 391867452$ (written in list, or one-line notation) contains $\sigma = 51342$, as can be seen by considering the subsequence 91672 ($= \pi(2), \pi(3), \pi(5), \pi(6), \pi(9)$). A *permutation class* is a downset (or hereditary property) of permutations under this order; thus if $\mathcal{C}$ is a permutation class, $\pi \in \mathcal{C}$, and $\sigma \leqslant \pi$ then $\sigma \in \mathcal{C}$. The generating function for a class $\mathcal{C}$ of permutations is $\sum_{\pi \in \mathcal{C}} x^{|\pi|}$, where, generally, this sum includes the empty permutation.

The importance of simple permutations to the study of permutation classes was first observed by Albert and Atkinson:

Theorem 2 (Albert and Atkinson [1]; see Brignall, Huczynska, and Vatter [5] for refinements)**.** *Every permutation class with only finitely many simple permutations has an algebraic generating function.*

After proving Proposition 1, we show how that proposition and Theorem 2 can be used to give a short proof of the following result.

Theorem 3 (Bóna [2] and independently Mansour and Vainshtein [7])**.** *For any integer r, the class of all permutations containing at most r copies of 132 has an algebraic generating function.*

We need a bit more notation before giving our proof of Proposition 1. The permutation π of length n is said to be *sum indecomposable* or *connected* if there is no integer $2 \leqslant i \leqslant n-1$ such that $\pi(\{1, 2, \ldots, i\}) = \{1, 2, \ldots, i\}$. Note that simple permutations are trivially sum indecomposable.

Proposition 4. *Suppose that π is a sum indecomposable permutation of length at least 4 in which $\pi(1) < \pi(n)$. Then every entry of π either participates in or separates (i.e., lies vertically or horizontally amongst) a copy of 2413 or 3142.*

Proof. Let p_1 denote the first entry of π, and choose p_2 to be the rightmost entry of π which lies below p_1. As $\pi(1) < \pi(n)$, p_2 is not the last entry of π. As π is not sum decomposable, there is at least one entry above p_1 which separates p_1 from p_2 (i.e., it lies horizontally between p_1 and p_2); let p_3 denote the greatest entry. Similarly, there is at least one entry to the right of p_2 which separates p_3 from p_1 (i.e., it lies vertically between p_1 and p_3); let p_4 denote the rightmost such entry. Note that p_1, p_2, p_3, and p_4 form a copy of 2413, so we are done if p_4 is the last entry of π. Otherwise we continue this process, alternatively choosing p_{2m+1} as the greatest entry of π which separates p_{2m-1} from p_{2m} and p_{2m+2} as the rightmost entry which separates p_{2m+1} from p_{2m}. This process will terminate when p_{2m+2} is the rightmost entry of π, at which point we have that every point of π is part of or lies horizontally amongst the sequence $p_1, p_2, \ldots, p_{2m+2}$. The proposition is then proved by observing that every four consecutive points of this sequence form a copy of 2413 or 3142. □

In the language of Brignall, Huczynska, and Vatter [4], the proof of Proposition 4 constructs a "proper right-reaching pin sequence". We are now ready to prove the main result.

Proof of Proposition 1. Let π be a simple permutation of length at least 4 and consider an arbitrarily entry $\pi(k)$. By symmetry we may assume that $\pi(1) < \pi(n)$ so since π is sum indecomposable, we are done by the previous proposition except in the case where $\pi(k)$ separates a copy of 2413 or 3142. By symmetry we may suppose $\pi(k)$ separates a copy of 2413. We therefore have the situation depicted in Figure 1. Since $\pi(k)$ must separate this copy of 2413, it cannot lie in any of the four corner cells, and in every other cell it could lie

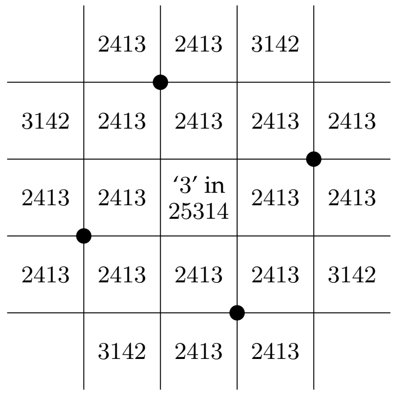

Figure 1: The situation in the proof of Proposition 1.

in, it gives rise to either a copy of 2413 or 3412 or, if it lies in the center square, it forms the '3' in a copy of 25314, completing the proof. ☐

In the statement of our next result, we say that the entry $\pi(k)$ of π is a *left-to-right minima* if there are no smaller entries to the left of $\pi(k)$.

Proposition 5. *If π is a simple permutation of length at least 4, then every entry of π which is not a left-to-right minima is contained in a copy of 132.*

Proof. Consider an arbitrary element $\pi(k)$ of the simple permutation π of length at least 4. By Proposition 1, $\pi(k)$ is contained in a copy of 2413, 3142, or plays the role of the '3' in a copy of 25314 or 41352. Since the 3 in both 24314 and 41352 is contained in a copy of 132, it suffices to consider the cases where $\pi(k)$ is contained in 2413 or 3142. Furthermore, the entries 243 of 2413 and the entries 142 of 3142 form copies of 132, so we are done except when $\pi(k)$ plays the role of a '1' in 2413 or a '3' in 3142. By symmetry, we need only consider the case where $\pi(k)$ plays the role of a '1' in a copy of 2413; let this copy of 2413 consist of the entries $\pi(i), \pi(j), \pi(k), \pi(\ell)$.

If π contains an entry below $\pi(k)$ and to the left of $\pi(j)$, then this entry together with $\pi(j)$ and $\pi(k)$ give rise to a copy of 132, and we are done. Thus we may assume that there are no entries in this region. Furthermore, if $\pi(k)$ is not a left-to-right minima of π, then there must be an entry of π below and to the left of $\pi(k)$; let the leftmost such entry be $\pi(a)$ and the least such entry be $\pi(b)$ (we allow the possibility that these two entries are identical). By our choice of $\pi(a)$ and $\pi(b)$, π contains no entries below $\pi(k)$ and to the left of $\pi(a)$, and it contains no entries below $\pi(b)$ and to the left of $\pi(k)$. We therefore have the situation depicted in Figure 2.

Now if π contains an entry horizontally between $\pi(a)$ and $\pi(k)$ and above $\pi(k)$ then this entry together with $\pi(a)$ and $\pi(k)$ would form a copy of 132, while if π contains an entry vertically between $\pi(b)$ and $\pi(k)$ and to the right of $\pi(k)$ then this entry together with $\pi(b)$ and $\pi(k)$ would form a copy of 132. If both of these regions are empty, however,

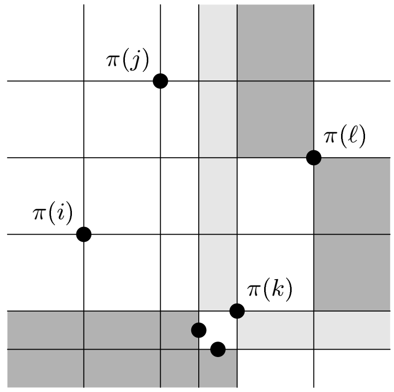

Figure 2: The situation in the proof of Proposition 5 when $\pi(k)$ is not a left-to-right minima. If π contains entries in any of the shaded regions then $\pi(k)$ is contained in a copy of 132; otherwise, π contains a nontrivial interval, and thus is not simple.

$\{\pi(a), \pi(b), \pi(k)\}$ forms a nontrivial interval in π, contradicting our assumption that π is simple. □

We can now quickly deduce Theorem 3.

Proof of Theorem 3. We need to establish that the class of permutations with at most r copies of 132 contains only finitely many simple permutations, as then Theorem 3 will follow from Theorem 2. By Proposition 5, if the simple permutation π of length at least 4 contains $3(r+1)$ entries which are not left-to-right minimas, then since each of these entries must participate in at least one copy of 132, π must contain at least $r+1$ copies of 132. It therefore suffices to bound the number of left-to-right minima in a simple permutation.

We claim that no simple permutation of length n can contain more than $2n/3$ left-to-right minima. This is perhaps easiest to see by considering it in a slightly different way. Since the left-to-right minima of a permutation form a decreasing subsequence, we claim that no simple permutation of length less than $3m/2$ can contain $m \cdots 21$. To see this, note that every adjacent pair $i+1, i$ in $m \cdots 21$ must be separated either horizontally or vertically by at least one entry, and that no simple permutation (of length at least 4) may begin with its largest entry or end with its least entry. Thus we need to add at least $m/2+2$ additional entries to $m \cdots 21$ in order to form a simple permutation. With the bound on the number of left-to-right minima in a simple permutation established, the theorem follows from our previous remarks. □

We note in closing that the proof of Theorem 3 we have given uses the notion of "simple extensions" of permutations. This concept is further studied by Brignall, Ruškuc, and Vatter [6], who prove that every permutation of length n has a simple extension with at most $\lceil (n+1)/2 \rceil$ additional entries.

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