# PROBLEMS AND CONJECTURES PRESENTED AT THE FIFTH INTERNATIONAL CONFERENCE ON PERMUTATION PATTERNS (UNIVERSITY OF ST ANDREWS, JUNE 11–15, 2007)

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## 1. A very brief introduction to permutation patterns

We say a permutation $\pi$ *contains* or *involves* the permutation $\sigma$ if deleting some of the entries of $\pi$ gives a permutation that is order isomorphic to $\sigma$, and we write $\sigma \leqslant \pi$. For example, 534162 contains 321 (delete the values 4, 6, and 2). A permutation *avoids* a permutation if it does not contain it.

This notion of containment defines a partial order on the set of all finite permutations, and the downsets of this order are called *permutation classes*. For a set of permutations $B$ define $\mathrm{Av}(B)$ to be the set of permutations that avoid all of the permutations in $B$. Clearly $\mathrm{Av}(B)$ is a permutation class for every set $B$, and conversely, every permutation class can be expressed in the form $\mathrm{Av}(B)$.

For the problems we need one more bit of notation. Given permutations $\pi$ and $\sigma$ of lengths $m$ and $n$, respectively, their *direct sum*, $\pi \oplus \sigma$, is the permutation of length $m+n$ in which the first $m$ entries are equal to $\pi$ and the last $n$ entries are order isomorphic to $\sigma$ while their *skew sum*, $\pi \ominus \sigma$, is the permutation of length $m+n$ in which the first $m$ entries are order isomorphic to $\pi$ while the last $n$ entries are equal to $\pi$. For example, $231 \oplus 321 = 231654$ and $231 \ominus 321 = 564321$.

## 2. Growth rates

We define the *growth rate* of a permutation class $\mathrm{Av}(B)$ as

$$\mathrm{gr}(\mathrm{Av}(B)) = \limsup_{n\to\infty} \sqrt[n]{|\,\mathrm{Av}(B) \cap S_n|},$$

where $S_n$ denotes the set of all permutations of length $n$. (This limit supremum is not known to be a limit except in special cases (e.g., when $|B| = 1$) where a supermultiplicativity argument applies, see Arratia [4]; it is, however, known to be finite so long as $B \neq \varnothing$ by the Marcus-Tardos Theorem [15].)

**Question 1** (contributed by Mike Atkinson)**.** *For all permutations $\beta$ and $k \geqslant 1$, do we have* $\mathrm{gr}(\mathrm{Av}(k \cdots 21, \beta)) = \mathrm{gr}(\mathrm{Av}(k \cdots 21, 1 \oplus \beta))$*?*

Subsequent research conducted at the University of Otago (and communicated to this author by Albert) provides a generalization of this question for $k = 3$: they proved that $\mathrm{gr}(\mathrm{Av}(321, \alpha \oplus \beta)) = \mathrm{gr}(\mathrm{Av}(321, \alpha \oplus 1 \oplus \beta))$ for all permutations $\alpha$ and $\beta$ (to get the $k = 3$ case of Problem 1, take $\alpha$ to be the empty permutation). It is also easy to see that

$$\mathrm{gr}(\mathrm{Av}(k \cdots 21, \alpha \oplus 1 \oplus \beta)) = \mathrm{gr}(\mathrm{Av}(k \cdots 21, \alpha \oplus 12 \oplus \beta))$$

for all $k$, $\alpha$, and $\beta$. (Communicated by Atkinson.)

For *principally based classes*, i.e., classes of the form $\mathrm{Av}(\beta)$ for a single permutation $\beta$, there have been a series of conjectures which we briefly recap. For any $\pi \in S_3$, the growth rate of $\mathrm{Av}(\pi)$ is 4, because the $\pi$-avoiding permutations are counted by the Catalan numbers. The growth rate of $\mathrm{Av}(12\ldots k)$ is $(k-1)^2$ by Regev [17]. It was an old conjecture that the growth rate of $\mathrm{Av}(\pi)$ is $(k-1)^2$ for all $\pi \in S_k$, but this was disproved by Bóna [9], who gave an exact enumeration of

Av(1342) which shows that its growth rate is 8. It is also tempting from this data to conjecture that the growth rate of $\mathrm{Av}(\pi)$ is always an integer, or at least rational; Bóna [11] disproved this by showing that the growth rate of Av(12453) is $9 + 4\sqrt{2}$.

**Question 2** (contributed by Miklós Bóna)**.** *Are all growth rates of principally based classes algebraic integers?*

Note that Albert and Linton [3] have constructed uncountably many permutation classes with different growth rates, and so it follows that there are permutation classes with non-algebaric growth rates (though they needn't be principally based). The obvious candidate to provide a negative answer to Question 2 is the class Av(1324), for which we know only that $\mathrm{gr}(\mathrm{Av}(1324)) > 9.47$ (proved by Albert, Elder, Rechnitzer, Westcott, Zabrocki [2], this bound disproved Arratia's earlier conjecture [4] that $\mathrm{gr}(\mathrm{Av}(\beta)) \leqslant (|\beta| - 1)^2$.)

## 3. Sorting

A stack is a last-in first-out linear sorting device with push and pop operations. The greedy algorithm for stack sorting a permutation $\pi = \pi(1)\pi(2)\dots\pi(n)$ goes as follows. First we push $\pi(1)$ onto the stack. Now suppose at some later stage that the letters $\pi(1), \dots, \pi(i-1)$ have all been either output or pushed on the stack, so we are reading $\pi(i)$. We push $\pi(i)$ onto the stack if and only if $\pi(i)$ is lesser than any element on the stack. Otherwise we pop elements off the stack until $\pi(i)$ is less than any remaining stack element and we push $\pi(i)$ onto the stack. For sorting with one stack this *greedy algorithm* is optimal, and its analysis quickly leads one to the conclusion that a single stack can sort precisely those permutations in Av(231).

West [22] considered the permutations that can be sorted by using the above greedy algorithm twice. He proved that the permutations sortable by this algorithm are those that avoid 2341 and in which every copy of 3241 and be extended by a single entry to a copy of 35241. However, this algorithm is *not* optimal, i.e., there are permutations that can be sorted by two stacks in series but cannot be sorted in this manner. In fact, the following question remains open:

**Question 3.** *Is it decidable in polynomial time (in $n$) if the permutation $\pi$ of length $n$ can be sorted by two stacks in series?*

Atkinson, Murphy, and Ruškuc [5] considered a sorting machine consisting of two stacks in series, subject to the restriction that the entries in each of the stacks must remain ordered. They presented an optimal algorithm for sorting with such a machine, showed that the class of sortable permutations is

$$\mathrm{Av}(\{2, 2k-1, 4, 1, 6, 3, 8, 5, \dots, 2k, 2k-3 : k \geqslant 2\}),$$

and constructed a bijection between this set and Av(1342). This suggests the following question.

**Question 4** (contributed by Miklós Bóna)**.** *Is there a natural sorting machine / algorithm which can sort precisely the class* Av(1342)*?*

For more information on stack sorting we refer the reader to Bóna's survey [10], and for more open problems we refer to the problems presented at Permutation Patterns 2005 [13].

## 4. Wilf-Equivalence

We say that the sets $B_1$ and $B_2$ are *Wilf-equivalent* if $|\,\mathrm{Av}_n(B_1)| = |\,\mathrm{Av}_n(B_2)|$ for all natural numbers $n$, that is, if $B_1$ and $B_2$ are equally avoided. Clearly natural symmetries of permutation classes give Wilf-equivalences, but many nontrivial Wilf-equivalences have been found. For example, it is a classic result that every permutation in $S_3$ is Wilf-equivalent to every other permutation in $S_3$. Another example is given above: $\{2, 2k-1, 4, 1, 6, 3, 8, 5, \dots, 2k, 2k-3 : k \geqslant 2\}$ is Wilf-equivalent to 1342. To date, the following problem remains open:

**Problem 5.** *Find necessary and sufficient conditions for two permutations to be Wilf-equivalent.*

There has been considerable work on the sufficient conditions front:

- Stankova [20] constructed a bijection between the generating trees of Av(4132) and Av(3142), establishing that 4132 and 3142 are Wilf-equivalent,
- Stankova and West [19] proved that $231 \ominus \beta$ and $132 \ominus \beta$ are Wilf-equivalent for all permutations $\beta$,
- Backelin, West, and Xin [8] proved that, for all $k$ and $\beta$, $12 \cdots k \oplus \beta$ and $k \cdots 21 \oplus \beta$ are Wilf-equivalent (thus generalizing the results of West [21] and Babson and West [7]).

These results, together with computer calculations, complete the classification of singleton Wilf-equivalences up to and including permutations of length 7.

Necessary conditions, on the other hand, have so far been lacking, and the only general way to show that $\alpha$ and $\beta$ are *not* Wilf-equivalent is to compute $|\operatorname{Av}_n(\alpha)|$ and $|\operatorname{Av}_n(\beta)|$ until they disagree.

More general necessary conditions would require results on permutations of small "codimension" that contain $\alpha$ and $\beta$. More precisely, consider then function

$$g_k(\beta) = |\{\text{permutations } \pi \text{ of length } |\beta| + k \text{ which contain } \beta\}|.$$

We have

$$|\operatorname{Av}_n(\beta)| = n! - g_{n-|\beta|}(\beta),$$

so $\alpha$ and $\beta$ are not Wilf-equivalent if and only if $g_k(\alpha) \neq g_k(\beta)$ for some $k$.

For a permutation $\beta$ of length $k$, Ray and West [16] show that

$$g_1(\beta) = k^2 + 1$$

(which they attribute to Bloomberg [unpublished, 1990]) and

$$g_2(\beta) = \left(k^4 + 2k^3 + k^2 + 4k + 4 - 2j\right)/2$$

for some $0 \leqslant j \leqslant k-1$.

**Problem 6** (contributed by Vince Vatter)**.** *Express the quantity $j$ above in terms of statistics of $\beta$.*

**Problem 7** (contributed by Vince Vatter)**.** *Find a formula for $g_3(\beta)$.*

## 5. Long Subsequences

The longest increasing subsequence ($LIS$) problem asks (in our language) for the greatest $k$ such that a given permutation $\pi$ of length $n$ contains $12 \cdots k$. The fastest algorithm for computing the $LIS$ is $O(n \log n)$, due to Schensted [18], and this bound is essentially best possible. Albert et al. [1] studied the longest $X$-subsequence ($LXS$) problem, which asks, for a set $X$ of permutations, for the longest member of $X$ that a permutation $\pi$ of length $n$ contains. They presented $O(n^2 \log n)$ algorithms to compute the $LXS$ for all cases where $X = \operatorname{Av}(B)$ and $B \subset S_3$ except for the case $X = \operatorname{Av}(231)$, where they presented a dynamic programming algorithm with runtime $O(n^5)$.

**Problem 8** (presented by Michael Albert)**.** *Give a faster algorithm for the $X = \operatorname{Av}(231)$ case of the $LXS$ problem.*

## 6. Generalized Patterns

A *generalized* (also known as *blocked*, *gapped*, or *Babson-Steingrímsson*, after their inventors [6]) pattern is one including dashes indicating the entries that need not occur consecutively (recall that no entries need occur consecutively in the normal pattern-containment order). For example, 24135 contains only one copy of 1-23, namely 235; the entries 245 do not form a copy of 1-23 because 4 and 5 are not adjacent.

In some cases, e.g., 2-1 and 21 or 2-31 and 2-3-1, avoiding a generalized pattern is equivalent to avoiding the underlying classical pattern. This leads us to:

**Question 9** (contributed by Einar Steingrímsson). *For which generalized patterns $\beta$ is avoiding $\beta$ equivalent to avoiding the underlying classical pattern? For which pairs of sets $B_1$, $B_2$ of generalized patterns is avoiding $B_1$ and $B_2$ equivalent?*

See Hardarson [14] for some recent progress on these questions.

## 7. Permutations of Special Form

Let $D_n$ denote the set of permutation matrices of dimension $2n-1 \times 2n-1$ where ones can appear either on or below the main diagonal or on or below the main diagonal of the $n \times n$ submatrix in the upper right-hand corner. For example, the cells where ones are allowed for $n = 5$ are denoted by $*$ in the matrix below.

$$\begin{pmatrix}
* & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & * & * & 0 & 0 \\
* & * & * & 0 & 0 & * & * & * & 0 \\
* & * & * & * & 0 & * & * & * & * \\
* & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & 0 \\
* & * & * & * & * & * & * & * & *
\end{pmatrix}$$

Burstein and Stromquist [12] have proved that $|D_n|$ is the $n$th Genocchi number.

**Problem 10** (contributed by Alex Burstein). *Give a bijective proof that $D_n$ is enumerated by the Genocchi numbers.*

Finally, we return to Problem 1. The proof in the $k = 3$ case relies on the following notion: the permutation $\pi$ is said to be $k$-rigid if every entry of $\pi$ participates in some copy of $k \cdots 21$. Asymptotically, 4/9th of the permutations in Av(321) are 2-rigid. However, it is not known if this behavior continues:

**Problem 11** (contributed by Mike Atkinson). *Prove that a positive fraction of the permutations in* $\mathrm{Av}((k+1)\cdots 21)$ *are $k$-rigid.*

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