

**PROBLEMS AND CONJECTURES PRESENTED AT THE THIRD
INTERNATIONAL CONFERENCE ON PERMUTATION PATTERNS
(UNIVERSITY OF FLORIDA, MARCH 7–11, 2005)**

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1. A VERY BRIEF INTRODUCTION TO PERMUTATION PATTERNS

A *permutation* is an arrangement of a finite number of distinct elements of a linear order, for example, $e, \pi, 0, \sqrt{2}$ and 3412. Two permutations are *order isomorphic* if they have the same relative ordering. We say a permutation τ *contains* or *involves* the permutation β if deleting some of the entries of τ gives a permutation that is order isomorphic to β , and we write $\beta \leq \tau$. For example, 534162 (when permutations contain only single digit natural numbers we suppress the commas) contains 321 (delete the values 4, 6, and 2). A permutation *avoids* a permutation if it does not contain it.

For a set of permutations B define $\text{Av}(B)$ to be the set of permutations that avoid all of the permutations in B and let $s_n(B)$ denote the number of permutations of length n in $\text{Av}(B)$. A set of permutations or class C is *closed* if $\pi \in C$ and $\sigma \leq \pi$ implies $\sigma \in C$. Therefore $\text{Av}(B)$ is a class for every set of permutations B and every permutation class can be written as $\text{Av}(B)$ for some set B . An *antichain* of permutations is a set of permutations such that no permutation contains another. We call a set of permutations B the *basis* of a class of permutations C if $C = \text{Av}(B)$ and B is an antichain.

For more information:

- Bóna's book [10] gives a good introduction to permutation patterns,
- Kitaev and Mansour have a survey [13] on the topic,
- Volume 9 (2) of the *Electronic Journal of Combinatorics* is devoted to permutation patterns,
- *Advances in Applied Mathematics* will be publishing a special issue on permutation patterns to accompany this conference.

2. EXACT ENUMERATION

Background: A sequence s_n is said to be P -recursive (which is short for *polynomially recursive*) if there are polynomials p_0, p_1, \dots, p_k so that

$$p_k(n)s_{n+k} + p_{k-1}(n)s_{n+k-1} + \dots + p_0(n)s_n = 0$$

For example, $s_n = n!$ is P -recursive since $s_{n+1} - (n+1)s_n = 0$. A sequence is P -recursive if and only if its generating function is D -finite, meaning that its derivatives span a finite dimensional vector space over $\mathbb{C}(x)$. More information on P -recursive sequences can be found in Stanley [24] and Zeilberger [28].

Let $\pi_1, \pi_2, \dots, \pi_k$ be permutations and r_1, r_2, \dots, r_k positive integers. Noonan and Zeilberger [17] conjectured that the number of n -permutations with exactly r_i copies of π_i for

each $i \in [k]$ forms a P -recursive sequence in n . Atkinson [3] showed that this is equivalent to the seemingly weaker conjecture¹ that $s_n(B)$ is P -recursive for all finite sets of patterns B .

For $B = \{1324\}$, Marinov and Radoičić [15] used a generating tree approach to find the first 20 terms. Recent work by Albert, Elder, Rechnitzer, Westcott and Zabrocki [1] gives more terms for the sequence $s_n(1324)$ (another six or so) which suggest this sequence is not well behaved.

Conjecture (Doron Zeilberger): The Noonan-Zeilberger conjecture is false, and, in particular, the sequence $s_n(1324)$ is not P -recursive.

In fact, Zeilberger made the stronger claim that “not even God knows $a_{1000}(1324)$.” He² suggests that we replace the Noonan-Zeilberger conjecture with the following research program:

Problem (Doron Zeilberger): Find necessary and sufficient conditions characterizing the classes whose generating functions are:

- rational,
- algebraic,
- D -finite,
- ...

Currently we have only one result of this type: Kaiser and Klazar [12] characterize the classes with polynomial enumeration.

One can also consider pattern avoiding compositions of an integer. Let

$$\mathbf{a} = (a_1, a_2, \dots, a_k)$$

denote a composition of n , so $a_1 + a_2 + \dots + a_k = n$, the a_i s need not be distinct, and order matters. We say that \mathbf{a} contains the permutation β if it contains a subsequence that is order isomorphic to β .

\$201 Conjecture (Herb Wilf): The number of 123-avoiding compositions of n into positive parts is not a P -recursive sequence.

Savage and Wilf [19] have found the generating function for this sequence:

$$\sum_{i \geq 1} \frac{1}{1 - x^i} \prod_{j \neq i} \left\{ \frac{1 - x^i}{(1 - x^{j-i})(1 - x^i - x^j)} \right\}.$$

Zeilberger pointed out that this function appears to have infinitely many singularities, indicating that it is not D -finite.

¹Some have called this Gessel’s conjecture, although to be fair to Gessel, all he wrote in [11] was that

Another possible candidate for P -recursiveness is the problem of counting permutations (or more generally sequences) with forbidden subsequences defined by inequalities, for example permutations $a_1 a_2 \dots a_n$ of $\{1, 2, \dots, n\}$ with no subsequence $a_i a_j a_k$ satisfying $a_i < a_k < a_j$.

²Zeilberger, not God

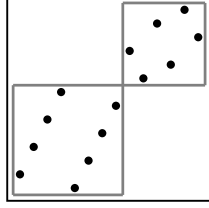


FIGURE 1. The plot of $24681357 \oplus 315264 = 2, 4, 6, 8, 1, 3, 5, 7, 11, 9, 13, 10, 14, 12$.

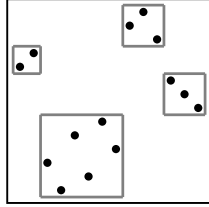


FIGURE 2. The plot of $3142[12, 315264, 231, 321] = 10, 11, 3, 1, 5, 2, 6, 4, 13, 14, 12, 9, 8, 7$.

3. COMPARISON OF PATTERNS

Background: We say that the sets B_1 and B_2 are *Wilf-equivalent*, and write $B_1 \sim B_2$, if $s_n(B_1) = s_n(B_2)$ for all natural numbers n , that is, if B_1 and B_2 are equally avoided. For example, it is a classic result that every permutation in S_3 is Wilf-equivalent to every other permutation in S_3 , or in other words, S_3 contains only one Wilf-equivalence class. It is a much more recent result (due to Stankova [21, 23] and West [27]) that S_4 contains three Wilf-equivalence classes. Although it was not raised during the problem session, we can not resist mentioning the following tantalizing problem.

Problem: Find necessary and sufficient conditions for two permutations to be Wilf-equivalent.

Many had thought that the permutations of a certain length could be ordered according to avoidance, modulo Wilf-equivalence. Stankova and West [22] were the first to find a counterexample. They observed that $s_n(53241) < s_n(43251)$ for all $n \leq 12$ while $s_{13}(53241) > s_{13}(43251)$. Still, they conjectured that the permutations of a certain length could be asymptotically ordered according to avoidance, modulo Wilf-equivalence, where we say that π is asymptotically more avoidable than σ if $s_k(\pi) > s_k(\sigma)$ for all sufficiently large k .

The conjectures in this section both attempt to understand these orderings, but before stating them we need some additional notation.

Given two permutations $\pi \in S_m$ and $\sigma \in S_n$, we define the *direct sum* (or simply *sum*) of π and σ , written $\pi \oplus \sigma$ by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } i \in [m], \\ \sigma(i - m) + m & \text{if } i \in [m + n] \setminus [m]. \end{cases}$$

Figure 1 shows an example.

A permutation is said to be *layered* if it is the direct sum of some number of decreasing permutations. For example, $21 \oplus 321 \oplus 1 = 215436$ is a layered permutation.

Let π be a permutation of length k and let $\sigma_1, \sigma_2, \dots, \sigma_k$ also be permutations. We define the *inflation* of π by $\sigma_1, \sigma_2, \dots, \sigma_k$ to be the permutation obtained by replacing the i th element of π by a block order isomorphic to σ_i so that the blocks themselves are order isomorphic to π . We write the resulting permutation as $\pi[\sigma_1, \sigma_2, \dots, \sigma_k]$. An example is shown in Figure 2. Direct sums fit into this framework as well, since $\pi \oplus \sigma = 12[\pi, \sigma]$.

Conjecture (Alex Burstein): For any permutations $\sigma_1, \sigma_2, \sigma_3$, and any natural number n ,

- (1) $s_n(132[\sigma_1, \sigma_2, \sigma_3]) < s_n(123[\sigma_1, \sigma_2, \sigma_3])$,
- (2) $s_n(312[\sigma_1, \sigma_2, \sigma_3]) < s_n(123[\sigma_1, \sigma_2, \sigma_3])$,
- (3) $s_n(123[\sigma_1, 12 \dots t, \sigma_3]) < s_n(123[\sigma_1, t \dots 21, \sigma_3])$ for any $t \geq 2$.

Conjecture (Alex Burstein): For any nonlayered permutation $\pi \in S_k$, permutations $\sigma_1, \sigma_2, \dots, \sigma_k$, and natural numbers n ,

$$s_n(\pi[\sigma_1, \sigma_2, \dots, \sigma_k]) < s_n(12 \dots k[\sigma_1, \sigma_2, \dots, \sigma_k]).$$

Of course, we don't have a proof of this latter conjecture even in the special case $\sigma_1 = \sigma_2 = \dots = \sigma_k = 1$, and if we did, we would have a different proof of the Stanley-Wilf conjecture via Regev [18].

Bóna's conjecture takes a little more preparation. To any permutation π on $[n]$ we associate a poset P_π . The elements of this poset are the integers 1 through n (which we will think of as the values of π , although we could equivalently think of them as being the indices of π). We will denote this partial order by \preceq , and write $i \preceq j$ if $i \leq j$ (as integers) and if i occurs before j in π (reading from left to right). Figure 3 shows two examples.

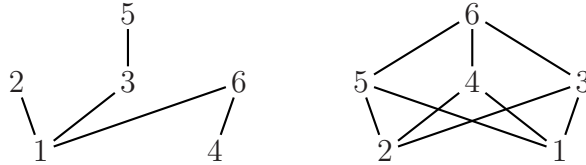


FIGURE 3. The posets corresponding to 416352 (left) and 215436 (right)

These posets are ranked, and the rank of i in P_π is the length of the longest increasing subsequence of π that ends in i . Given a poset P with rank function r , we let $\text{conv}(P)$ denote the *convex hull* of P . The poset $\text{conv}(P)$ is defined on the same set of elements as P , but in $\text{conv}(P)$, $i \preceq_{\text{conv}} j$ if and only if $r(i) \leq r(j)$ (in P). Figure 4 shows the convex hulls of the posets from Figure 3. Note that $\text{conv}(P_{215436}) = P_{215436}$ and that $\text{conv}(P_{416352}) \cong P_{215436}$. Indeed, these are not accidents. For any layered permutation π , $\text{conv}(P_\pi) = P$, and for any permutation π there is a unique layered permutation, which we refer to as $\text{conv}(\pi)$, such that $P_\pi \cong P_{\text{conv}(\pi)}$.

Conjecture (Miklós Bóna): For any permutation π and natural number n , $s_n(\pi) \leq s_n(\text{conv}(\pi))$.

Like Burstein's conjecture, Bóna's conjecture would give another proof of the Stanley-Wilf conjecture, via Bóna [8].

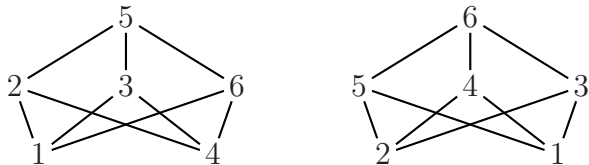


FIGURE 4. The convex hulls of the posets from Figure 3

4. GROWTH RATES

Background: Let X be a set of permutations of varying lengths. We define the *upper growth rate* of X to be

$$\overline{\text{gr}}(X) = \limsup_{n \rightarrow \infty} \sqrt[n]{|X \cap S_n|}$$

and the *lower growth rate* of X to be

$$\underline{\text{gr}}(X) = \liminf_{n \rightarrow \infty} \sqrt[n]{|X \cap S_n|}.$$

The Marcus-Tardos Theorem [14], formerly the Stanley-Wilf conjecture, implies that these limits are finite if X is a closed class and X is not the set of all permutations. For this reason some refer to $\text{gr}(\text{Av}(B))$ as the “Stanley-Wilf limit of $\text{Av}(B)$.”

Of course it would be natural to define the *growth rate* of X as

$$\text{gr}(X) = \lim_{n \rightarrow \infty} \sqrt[n]{|X \cap S_n|},$$

but this limit is not known to exist in general. In one important special case, Arratia [2] used Fekete’s Lemma to show that $\text{gr}(\text{Av}(\pi))$ always exists, but his argument does not generalize to the case of $\text{Av}(B)$ where $|B| \geq 2$. The results of Kaiser and Klazar [12] show that $\text{gr}(C)$ exists whenever C is a closed class with $\overline{\text{gr}}(C) \leq 2$. Indeed, their results show that if $\overline{\text{gr}}(C) < 2$ for a closed class C then there is some $k \geq 1$ so that $\text{gr}(C)$ is the largest positive real root of $x^k - x^{k-1} - \dots - x - 1$.

For any $\pi \in S_3$, the growth rate of $\text{Av}(\pi)$ is 4, because the π -avoiding permutations are counted by the Catalan numbers. The growth rate of $\text{Av}(12 \dots k)$ is $(k-1)^2$ by Regev [18]. It was an old conjecture that the growth rate of $\text{Av}(\pi)$ is $(k-1)^2$ for all $\pi \in S_k$, but this was disproved by Bóna [7], who showed that the growth rate of $\text{Av}(1342)$ is 8. It is also tempting from this data to conjecture that the growth rate of $\text{Av}(\pi)$ is always an integer; Bóna [5] disproved this also by showing that the growth rate of $\text{Av}(12453)$ is $9 + 4\sqrt{2}$.

The only permutation of length four for which the growth rate of $\text{Av}(\pi)$ is not known is 1324. Albert, Elder, Rechnitzer, Westcott, Zabrocki [1] recently showed that this growth rate is at least 9.35, thereby disproving a conjecture of Arratia.³ AERWZ postulate the actual growth rate for $\text{Av}(\pi)$ lies between 11 and 12. This brings us to

\$200 Problem (Doron Zeilberger): Improve on the AERWZ lower bound for $\text{gr}(\text{Av}(1324))$ using the Maple package WILFPLUS [26].

Conjecture (Vince Vatter): For any finite set of patterns B , the growth rate of $\text{Av}(B)$ exists.

³Arratia [2] had conjectured the growth rate of $\text{Av}(\pi)$ is at most $(k-1)^2$ for any $\pi \in S_k$.

Problem (Michael Albert): How quickly can an antichain of permutations grow? More precisely, how large can $\overline{\text{gr}}(X)$ be for an antichain X ?

Albert has an example of an antichain for which this limit is approximately 2.24.

5. STACK SORTING

Background: A stack is a last-in first-out linear sorting device with push and pop operations. The greedy algorithm for stack sorting a permutation $\pi = \pi(1)\pi(2)\dots\pi(n)$ goes as follows. First we push $\pi(1)$ onto the stack. Now suppose at some later stage that the letters $\pi(1), \dots, \pi(i-1)$ have all been either output or pushed on the stack, so we are reading $\pi(i)$. We push $\pi(i)$ onto the stack if and only if $\pi(i)$ is lesser than any element on the stack. Otherwise we pop elements off the stack until $\pi(i)$ is less than any remaining stack element and we push $\pi(i)$ onto the stack. This produces a permutation $s(\pi)$. A permutation is *West t -stack sortable* if $s^t(\pi)$ is the identity permutation.

Of course other stack sorting algorithms are possible, and for more than one stack, the iterated greedy algorithm described above is not the optimal algorithm (see, for example, Smith [20]). By allowing any sorting algorithm we reach the definition of *general t -stack sortability*. In addition to the other general references on permutation patterns, Bóna [9] provides a survey of stack sorting.

This more general sorting machine has “complete lookahead,” meaning that it can make its choices based on looking at the entire permutation. Julian West suggested that it might be interesting to study machines that can only see the next k letters of the permutations at a time.

Problem (Miklós Bóna): For which t, n is the number of West t -stack sortable permutations of length n an odd number? So far we know

- one stack: odd if and only if $n = 2^k - 1$ since 1-stack sortable permutations are counted by the Catalan numbers,
- two stacks: odd quite frequently; more, precisely, this number is odd if and only if the binary expansion of n does not contain two 1’s in consecutive positions and ends in a 1; the number of such values of n between 1 and 2^m is the m th Fibonacci number. Bóna [6] gives a combinatorial proof,
- three stacks: numerical evidence of Oliver Guibert: odd for $n=1,9$.

Thus with one stack odd numbers show up very rarely, with two stacks the numbers are quite often odd, but for three stacks odd numbers are again rare. Bóna conjectures that this pattern continues. He provided the following motivation: 1-stack sortable n -permutations can be described by a codeword on two letters of length $2n$ (using one letter for “in” and another for “out”); 2-stack sortable n -permutations can be described by a codeword on three letters of length $3n$ (using letters for “into first stack,” “from first to second stack,” and “out of second stack”). In the same manner, t -stack sortable n -permutations can be described by a codeword on $t+1$ letters of length $(t+1)n$.

Bruce Sagan pointed out that there may be a connection to trees that would help establish Bóna’s conjecture.

Problem (Mikós Bóna): In particular, the number of West 2-stack sortable permutations on $4k+3$ elements is always even. Give a combinatorial proof.

Problem (Mike Atkinson): Can one decide in polynomial time whether an input permutation is sortable by a general t -stack sorting machine, or is this problem NP-complete?

Note that this problem does lie in NP, because one can verify in polynomial time whether a permutation can be sorted in some particular set of steps. If the basis for general 2-stack sortable permutations were finite, then Atkinson's problem could be done in polynomial time, but Atkinson, Murphy, and Ruškuc [4] conjecture that this basis is finite.

Problem (Steve Waton): What is the shortest permutation that can not be sorted by a general 3-stack sorting machine?

Tarjan [25] gave upper and lower bounds for the shortest permutation that can't be sorted by a t -stack machine, but they don't appear to be sharp. Murphy [16] shows that the shortest permutations that can't be sorted by a general 2-stack sorting procedure have length 7, and there are 22 of them. For three stacks, Elder and Waton have wagered a beer on the problem, with Elder guessing that the shortest unsortable permutation is of length 15 and Waton betting on 22.

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