

# A SHARP BOUND FOR THE RECONSTRUCTION OF PARTITIONS

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Answering a question of Cameron, Pretzel and Siemons proved that every integer partition of  $n \geq 2(k+3)(k+1)$  can be reconstructed from its set of  $k$ -deletions. We describe a new reconstruction algorithm which lowers this bound to  $n \geq k^2 + 2k$  and present examples showing that this bound is best possible.

Analogues and variations of Ulam's notorious graph reconstruction conjecture have been studied for a variety of combinatorial objects, for instance words (see Schützenberger and Simon [2, Theorem 6.2.16]), permutations (see Raykova [4] and Smith [5]), and compositions (see Vatter [6]), to name a few.

In answer to Cameron's query [1] about the partition context, Pretzel and Siemons [3] proved that every partition of  $n \geq 2(k+3)(k+1)$  can be reconstructed from its set of  $k$ -deletions. Herein we describe a new reconstruction algorithm that lowers this bound, establishing the following result, which Negative Example 2 shows is best possible.

**Theorem 1.** *Every partition of  $n \geq k^2 + 2k$  can be reconstructed from its set of  $k$ -deletions.*

We begin with notation. Recall that a *partition of  $n$* ,  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ , is a finite sequence of nonincreasing integers whose sum, which we denote  $|\lambda|$ , is  $n$ . The *Ferrers diagram* of  $\lambda$ , which we often identify with  $\lambda$ , consists of  $\ell$  left-justified rows where row  $i$  contains  $\lambda_i$  cells. An *inner corner* in this diagram is a cell whose removal leaves the diagram of a partition, and we refer to all other cells as *interior cells*.

We write  $\mu \leq \lambda$  if  $\mu_i \leq \lambda_i$  for all  $i$ ; another way of stating this is that  $\mu \leq \lambda$  if and only if  $\mu$  is contained in  $\lambda$  (here identifying partitions with their diagrams). If  $\mu \leq \lambda$ , we write  $\lambda/\mu$  to denote the set of cells which lie in  $\lambda$  but not in  $\mu$ . We say that the partition  $\mu$  is a  *$k$ -deletion* of the partition of  $\lambda$  if  $\mu \leq \lambda$  and  $|\lambda/\mu| = k$ .

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Recall that this order defines a lattice on the set of all finite partitions, known as *Young's lattice*, and so every pair of partitions has a unique *join* (or *least upper bound*)

$$\mu \vee \lambda = (\max\{\mu_1, \lambda_1\}, \max\{\mu_2, \lambda_2\}, \dots)$$

and *meet*

$$\mu \wedge \lambda = (\min\{\mu_1, \lambda_1\}, \min\{\mu_2, \lambda_2\}, \dots).$$

Finally, recall that the *conjugate* of a partition  $\lambda$  is the partition  $\lambda'$  obtained by flipping the diagram of  $\lambda$  across the NW-SE axis; it follows that  $\lambda'_i$  counts the number of entries of  $\lambda$  which are at least  $i$ .

Before proving Theorem 1 we show that it is best possible:

**Negative Example 2.** For  $k \geq 1$ , consider the two partitions

$$\begin{aligned} \mu &= (\underbrace{k+1, \dots, k+1}_k, k-1) \text{ and} \\ \lambda &= (\underbrace{k+1, \dots, k+1}_{k-1}, k, k). \end{aligned}$$

Note that no  $k$ -deletion of  $\mu$  can contain the cell  $(k, k+1)$  and that no  $k$ -deletion of  $\lambda$  can contain the cell  $(k+1, k)$ . Therefore every  $k$ -deletion of  $\mu$  and of  $\lambda$  is actually a  $(k-1)$ -deletion of

$$\mu \wedge \lambda = (\underbrace{k+1, \dots, k+1}_{k-1}, k, k-1),$$

so  $\mu$  and  $\lambda$  cannot be differentiated by their sets of  $k$ -deletions.

We are now ready to prove our main result.

*Proof of Theorem 1.* Suppose that we are given a positive integer  $k$  and a set  $\Delta$  of  $k$ -deletions of some (unknown) partition  $\lambda$  of  $n \geq k^2 + 2k$ . Our goal is to determine  $\lambda$  from this information. We begin by setting  $\mu = \bigvee_{\delta \in \Delta} \delta$ , noting that we must have  $\lambda \geq \mu$ . Hence if  $|\mu| = n$  then we have  $\lambda = \mu$  and we are immediately done, so we will assume that  $|\mu| < n$ .

First consider the case where  $\mu$  has less than  $k$  rows. Let  $r$  denote the bottommost row of  $\mu$  which contains at least  $k$  cells ( $r$  must exist because  $\mu$  has less than  $k$  rows and  $|\mu| \geq k^2 + k$ ). Thus the  $r$ th row of  $\lambda$  contains at least  $k$  cells as well, so there are  $k$ -deletions of  $\lambda$  in which the removed cells all lie at or below row  $r$ . Hence the first  $r-1$  rows of  $\lambda$  and  $\mu$  agree. Now note that  $\lambda$  has more than  $2k$  cells to the right of column  $k$ , so there are  $k$ -deletions of  $\lambda$  in which the removed cells all lie to the right of column  $k$ , and thus the first  $k$  columns of  $\lambda$  and  $\mu$  agree. This implies that  $\lambda$  and  $\mu$  agree on all rows below  $r$  (since these rows have less than  $k$  cells in  $\mu$ ) and so all cells of  $\lambda/\mu$  must lie in row  $r$ , uniquely determining  $\lambda$ , as desired. The case where  $\mu$  has less than  $k$  columns follows by symmetry.

We may now assume that  $\mu$  has at least  $k$  rows and  $k$  columns. Let  $r$  (resp.  $c$ ) denote the bottommost row (resp. rightmost column) containing at least  $k$  cells. Both  $r$  and  $c$  exist

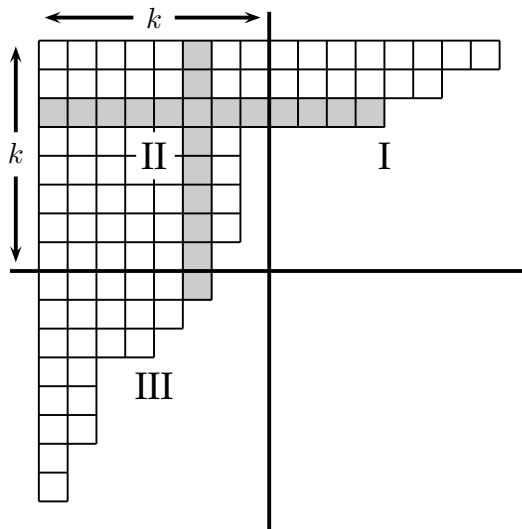


Figure 1: An example partition  $\mu$  from Case 1 of the proof of Theorem 1, divided into three quadrants. Here  $k = 8$ , and  $r$  and  $c$  appear shaded.

because  $\mu$  has at least  $k$  rows and columns. As before, we see that the first  $r - 1$  rows and the first  $c - 1$  columns of  $\lambda$  and  $\mu$  agree. We now consider three cases based on whether and where  $r$  and  $c$  intersect.

*Case 1:  $r$  and  $c$  intersect at an interior cell of  $\mu$ .* Suppose that  $r$  and  $c$  intersect at the cell  $(i, j)$ . It follows from the maximality of  $r$  and  $c$  that  $i, j < k$ , and thus the cell  $(k, k)$  does not lie in  $\mu$ . Since  $|\lambda| \geq k^2 + 2k$ , this implies that the cell  $(k, k)$  also fails to lie in  $\lambda$ . Therefore both  $\lambda$  and  $\mu$  can be divided into three quadrants, I, II, and III, as shown in Figure 1. Because Quadrant II of  $\lambda$  contain less than  $k^2$  cells,  $\lambda$  must have more than  $k$  cells in quadrant I or III. Hence there are also  $k$ -deletions of  $\lambda$  with more than  $k$  cells in quadrant I or III; suppose by symmetry that  $\lambda$  and  $\mu$  both have more than  $k$  cells in quadrant I.

Thus there are  $k$ -deletions of  $\lambda$  in which the removed cells are all chosen from quadrant I, so  $\lambda$  and  $\mu$  agree on all cells in quadrants II and III. This shows that  $r$  is also the bottom-most row of  $\lambda$  with at least  $k$  cells, and so  $\lambda/\mu$  contains no cells below row  $r$  in quadrant I. As we already know that  $\lambda$  and  $\mu$  agree on their first  $r - 1$  rows, we can therefore conclude that all cells of  $\lambda/\mu$  lie in row  $r$ , which allows us to reconstruct  $\lambda$  and complete the proof of this case.

*Case 2:  $r$  and  $c$  intersect at an inner corner of  $\mu$ .* It follows that  $r, c \geq k$ . Because  $\lambda$  and  $\mu$  agree to the left of column  $c$  and above row  $r$ , all cells of  $\lambda/\mu$  must lie below or to the right of  $(r, c)$ . However, the cell  $(r + 1, c + 1)$  cannot lie in  $\lambda$  because if it did then one could form a  $k$ -deletion of  $\lambda$  by removing only points lying to the right of column  $c$ , which would leave at least  $k$  cells in row  $r + 1$  and contradict the definition of  $r$ . This leaves only two possibilities for  $\lambda/\mu$ : the cells  $(r, c + 1)$  and  $(r + 1, c)$ . However, only one of these cells can be

added to  $\mu$  to produce a partition: if both could be added then both row  $r + 1$  and column  $c + 1$  of  $\lambda$  contain at least  $k$  cells, which implies that there are  $k$ -deletions of  $\lambda$  in which each contain at least  $k$  cells, contradicting the choice of  $r$  and  $c$ . This case therefore reduces to checking which one of the cells  $(r, c + 1)$  and  $(r + 1, c)$  can be added to  $\mu$  to produce a partition.

*Case 3:  $r$  and  $c$  do not intersect.* Suppose that the rightmost cell in row  $r$  is  $(r, j)$  and the bottommost cell in column  $c$  is  $(i, c)$ . If  $j < c - 1$  then because  $\lambda$  and  $\mu$  agree to the left of column  $c$ ,  $\lambda/\mu$  cannot contain any cells in or below row  $r$ , and we already have that  $\lambda$  and  $\mu$  agree above row  $r$ , so we are left with the conclusion that  $\lambda = \mu$ . By symmetry we are also done if  $i < r - 1$ , leaving us to consider the case where  $i = r - 1$  and  $j = c - 1$ . Again using the fact that  $\lambda$  and  $\mu$  agree above row  $r$  and to the left of column  $c$  (and the definitions of  $r$  and  $c$ ) we see that the only possibility for  $\lambda/\mu$  is  $(r, c)$ , completing the proof of this case and the theorem.  $\square$

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