Finitely-labeled generating trees and restricted permutations (extended abstract)

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The list of published papers on counting restricted permutations algorithmically is short: Zeilberger [6], published in 1998. In that paper Zeilberger presents an algorithm which, given a set of forbidden patterns Q and a maximum depth, attempts to find an enumeration scheme for the number of n-permutations avoiding Q. If the algorithm comes back with an answer, then we are lucky and have a fast recurrence. However, no proofs are offered for when the algorithm will work (except, of course, that when it does work it provides such a proof itself), and if the algorithm does not work, one can never know if increasing the maximum depth will help.

Our goal is three-fold. First, we would like to produce an algorithm for counting restricted permutations that we can prove will work in at least some situations. Secondly, we would like to have an algorithm that returns a generating function. Finally, we would like to have an algorithm that utilizes generating trees, which have been very helpful for humans attempting to count restricted permutations.

We were able to achieve these goals for a sufficiently narrow class of restrictions. Our algorithm turns out to be, up to symmetry, a special case of Zeilberger's algorithm.

Generating trees

A generating tree is a rooted, labeled tree such that the labels of the children of each node may are determined by the label of that node. Therefore we specify a generating tree by supplying the label of the root and a set of *succession rules*. For example, the complete binary tree may be given by

$$\begin{array}{lll} \text{Root:} & (2) & \\ \text{Rule:} & (2) & \rightsquigarrow \quad (2)(2). \end{array}$$

The other tree of interest to us is the *pattern-avoidance tree* $T(Q)$, whose nodes are the Q-avoiding permutations where the n-permutation $\sigma \in T(Q)$ is a child of $\pi \in T(Q)$ if σ can be obtained by inserting n into π. For example, $T(132, 231)$ is shown in Figure 1. The *active sites* (relative to Q) of $\pi \in \mathcal{A}_{n-1}(Q)$ are the positions i for which inserting n right before the

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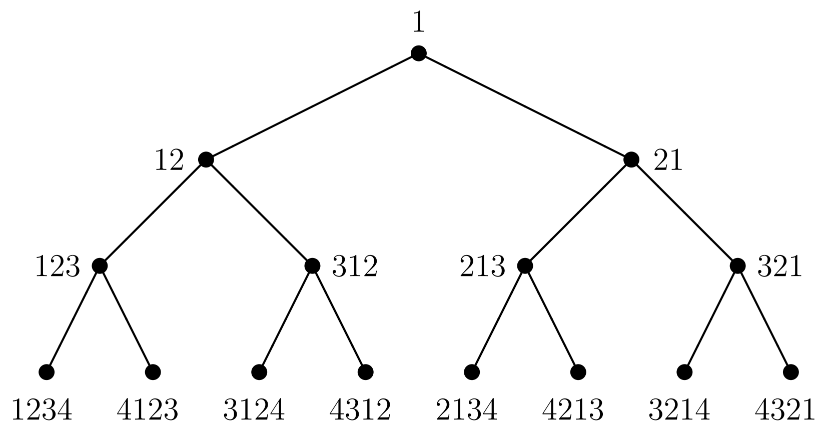

Figure 1: The first four levels of the pattern-avoidance tree $T(132, 231)$

ith element of π produces a Q-avoiding permutation. By convention, $n+1$ is an active site of the n-permutation π if appending n to the end of π produces a Q-avoiding permutation. An inactive site is any site that is not active. For example, the active sites of 461523 relative to $\{1243, 3421\}$ are 1, 2, 5, and 7, and the inactive sites are the complement: 3, 4, and 6.

Given a pattern-avoidance tree $T(Q)$, we would like to be able to find an isomorphic generating tree (here, and in what follows, we mean isomorphic as rooted trees). For example, let $\pi \in \mathcal{A}_{n-1}(132, 231)$. Then we may obtain a $\{132, 231\}$-avoiding permutation by inserting n either at the beginning or the end of π, but nowhere in between, so $T(132, 231)$ is isomorphic to the complete binary tree. The pattern-avoidance tree $T(123)$ is more complicated; West [5] showed that it is isomorphic to the generating tree given by

$$\begin{array}{lll} \text{Root:} & (2) & \\ \text{Rules:} & (j) & \rightsquigarrow \quad (2)(3)\ldots(j+1). \end{array}$$

Once we have found a generating tree isomorphic to $T(Q)$ it is in general not too difficult to enumerate Q-avoiding permutations. In particular, if this generating tree has only finitely many labels, as in the case of $T(132, 231)$, it is known that the generating function enumerating Q-avoiding permutations by length is rational.

Goal

We can now state our goal. Notice that there is a fundamental difference between $T(132, 231)$ and $T(123)$: whereas $T(132, 231)$ is isomorphic to a generating tree with only finitely many labels (indeed, only one!), $T(123)$ is not isomorphic to such a generating tree. We will produce an algorithm that will give the generating function for the number of permutations avoiding a set of patterns Q in the case that $T(Q)$ is isomorphic to a finitely-labeled generating tree. This might not seem very satisfying, but as a corollary of our algorithm, we will characterize the class of restrictions Q for which $T(Q)$ is isomorphic to a finitely-labeled generating tree.

One requirement for $T(Q)$ to be isomorphic to a finitely-labeled generating tree is that nodes of $T(Q)$ must have a bounded number of children. For nodes of $T(Q)$ to have a bounded number of children clearly Q must contain both a child of an increasing permutation (such as 132) and a child of a decreasing permutation (such as 231), because otherwise at least one of $12\dots n$ or $n\dots 21$ will have $n+1$ children for all n. In fact, Kremer and Shiu [2] showed that this is enough:

Theorem 1 *[2] The pattern-avoidance tree $T(Q)$ has bounded degrees if and only if Q contains both a child of an increasing permutation and a child of a decreasing permutation.*

The proof of Theorem 1 is a simple consequence of the result of Erdős and Szekeres [1] that long permutations have long monotone subsequences.

Preliminaries

The *reduction* of a word w of k distinct integers is the k-permutation $\mathrm{red}(w)$ obtained by replacing the smallest element of w by 1, the second smallest element by 2, and so on. So π contains a σ pattern if and only if π contains a (not necessarily contiguous) subword that reduces to σ.

If π is a node of $T(Q)$ let $T(Q;\pi)$ denote the subtree consisting of π and its descendants.

If x is an element of the permutation π, we write $\pi - x$ to denote the word formed from π by removing x. Thus our notation is such that we can write $461523 - 2 = 46153$ and be correct! We say that the element x is *removable* (relative to a set of restrictions Q) if it is adjacent to at most one active site. For example, $\pi(4)$ is removable if and only if at most one of 4 or 5 is an active site of π.

The most important property of removable elements, in fact the only property, is that if x is removable in π (as usual, relative to a set of restrictions Q), then $T(Q;\pi)$ embeds into $T(Q;\mathrm{red}(\pi - x))$. That is, given a permutation $\sigma \in T(Q;\mathrm{red}(\pi - x))$, there is at most one way we can insert x to get a permutation in $T(Q;\pi)$.

It sometimes happens that these two trees are isomorphic. If $T(Q;\pi) \cong T(Q;\mathrm{red}(\pi - x))$, *and* x is removable, then we say that x is *reinsertible* (again, relative to Q). This definition is similar to what Zeilberger [6] calls 'reversely deleteable.'

It is clearly possible to determine whether or not $x \in \pi$ is removable, but testing it for reinsertibility is not so clear. The next proposition shows that we can indeed perform this check, although in practice it might be expensive.

Proposition 2 *Let π be a Q-avoiding permutation of length at least two, and suppose that $x \in \pi$ is removable. Then x is reinsertible if and only if the number of nodes on level i of $T(Q;\pi)$ equals the number of nodes on level i of $T(Q;red(\pi - x))$ for all i from 1 to the length of the largest pattern in Q minus 1.*

The algorithm

Proposition 2 gives us the following approach for finding a generating tree isomorphic to $T(Q)$. If $1 \notin \mathcal{A}(Q)$ then our job is very easy (there are no Q-avoiding permutations!), so let us assume that 1 avoids Q. We start with a root node (1), a set $B = \{1\}$ of nodes that we have not checked for reinsertible elements, and a set $\mathcal{S} = \emptyset$ of succession rules. Now we pick a permutation $\pi \in B$ of minimum length and check it for reinsertible elements (we make the convention that the permutation 1 never has a reinsertible element).

First suppose that π does not have a reinsertible element, and say that its Q-avoiding children are $\pi_1, \pi_2, \ldots, \pi_t$. In this case we remove π from B, add its Q-avoiding children to B, and add the succession rule

$$(\pi) \rightsquigarrow (\pi_1)(\pi_2)\ldots(\pi_t)$$

to $\mathcal{S}$.

On the other hand, if π does have a reinsertible element x then we again remove π from B, but now we search through our set of succession rules $\mathcal{S}$ and replace each instance of (π) by the label we have given to the node $\mathrm{red}(\pi - x)$. In other words, whenever a node labeled by (π) would have been produced, now a node labeled by the same label as $\mathrm{red}(\pi - x)$ will be produced. This does not change the isomorphism type of the tree because

$$T(Q;\pi) \cong T(Q;\mathrm{red}(\pi - x)),$$

by definition of reinsertibility.

We repeat this process until (hopefully) $B = \emptyset$. If we ever reach this state then we know that the generating tree we have produced is isomorphic to $T(Q)$.

If $T(Q)$ does not have bounded degrees, then we will never reach a state where $B = \emptyset$, because in that case at least one of $12\ldots n$ or $n\ldots 21$ will not have a removable element, let alone a reinsertible element. Our central result, below, shows that if Q is finite and $T(Q)$ has bounded degrees then this procedure will terminate.

Theorem 3 *Let Q be a finite set of patterns. The pattern-avoidance tree $T(Q)$ is isomorphic to a finitely labeled generating tree if and only if Q contains both a child of an increasing permutation and a child of a decreasing permutation. Furthermore, if $T(Q)$ satisfies these conditions then the algorithm presented above will find a finitely-labeled generating tree isomorphic to $T(Q)$.*

Theorem 3 is proved by showing that all sufficiently long permutations have a reinsertible element with respect to Q.

Results of the algorithm

The algorithm presented here is implemented in the Maple package `FINLABEL` available from the author's homepage at `http://math.rutgers.edu/~vatter/`.

Among the results it has rediscovered are the classics from Simion and Schmidt [3]:

Q	o.g.f. for Q-avoiding permutations by length
$123, 213$	$\frac{x}{1-2x}$
$123, 231$	$\frac{x-x^2+x^3}{1-x^3}$
$123, 321$	$x + 2x^2 + 4x^3 + 4x^4$
$132, 231$	$\frac{x}{1-2x}$
$312, 231$	$\frac{x}{1-2x}$

Several results from West [4]:

Q	o.g.f. for Q-avoiding permutations by length
$123, 3214$	$\frac{x-x^2}{1-3x+x^2}$
$123, 3421$	$\frac{x-3x^2+5x^3-2x^4}{1-5x+10x^2-10x^3+5x^4-x^5}$
$123, 3241$	$\frac{x-3x^2+4x^3-x^4}{1-5x+9x^2-7x^3+2x^4}$
$123, 4321$	$x + 2x^2 + 5x^3 + 13x^4 + 25x^5 + 25x^6$
$132, 3214$	$\frac{x-2x^2+2x^3}{1-4x+5x^2-3x^3}$
$132, 3241$	$\frac{x+x^2}{1-3x+x^2}$
$132, 3421$	$\frac{x-3x^2+3x^2}{1-5x+8x^2-4x^3}$
$132, 4321$	$\frac{x-3x^2+5x^3-2x^4+x^5}{1-5x+10x^2-10x^3+5x^4-x^5}$
$312, 3214$	$\frac{x-x^2}{1-3x+x^2}$
$312, 3241$	$\frac{x-x^2}{1-3x+x^2}$
$312, 3421$	$\frac{x-x^2}{1-3x+x^2}$
$312, 4321$	$\frac{x-x^2}{1-3x+x^2}$

And cases with two patterns of length four, computed recently by Kremer and Shiu [2]:

Q	o.g.f. for Q-avoiding permutations by length
$1234, 3214$	$\frac{x-3x^2}{1-5x+4x^2}$
$1234, 3241$	$\frac{x-11x^2+54x^3-151x^4+268x^5-313x^6+234x^7-108x^8+29x^9-4x^{10}}{1-13x+74x^2-243x^3+510x^4-715x^5+678x^6-429x^7+173x^8-40x^9+4x^{10}}$
$1234, 3421$	$\frac{x-7x^2+24x^3-44x^4+62x^5-39x^6+32x^7-19x^8+4x^9}{1-9x+36x^2-84x^3+126x^4-126x^5+84x^6-36x^7+9x^8-x^9}$
$1234, 4321$	$x + 2x^2 + 6x^3 + 22x^4 + 86x^5 + 306x^6 + 882x^7 + 1764x^8 + 1764x^9$
$1243, 3214$	$\frac{x-5x^2+9x^3-8x^4+3x^5}{1-7x+17x^2-22x^3+13x^4-4x^5}$
$1243, 3241$	$\frac{x-9x^2+31x^3-49x^4+37x^5-14x^6+2x^7}{1-11x+47x^2-99x^3+109x^4-63x^5+18x^6-2x^7}$
$1243, 3421$	$\frac{x-9x^2+34x^3-64x^4+64x^5-28x^6+4x^7}{1-11x+50x^2-120x^3+160x^4-112x^5+32x^6}$
$1423, 3214$	$\frac{x-6x^2+12x^3-7x^4+2x^5}{1-8x+22x^2-25x^3+10x^4-2x^5}$
$1423, 3241$	$\frac{x-5x^2+8x^3-4x^4}{1-7x+16x^2-16x^3+4x^4}$
$4123, 3214$	$\frac{x-3x^2}{1-5x+4x^2}$

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