ALMOST AVOIDING PERMUTATIONS

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We investigate the notion of almost avoiding a permutation: π *almost avoids* β if one can remove a single entry from π to obtain a β-avoiding permutation.

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1. INTRODUCTION

The permutation π of length n, written in one-line notation as $\pi(1)\pi(2)\cdots\pi(n)$, is said to *contain* the permutation σ if π has a subsequence that is order isomorphic to σ, and each such subsequence is said to be an *occurrence* of σ in π or simply a σ pattern. For example, $\pi = 491867532$ contains $\sigma = 51342$ because of the subsequence $\pi(2)\pi(3)\pi(5)\pi(6)\pi(9) = 91672$. Permutation containment is easily seen to be a partial order on the set of all (finite) permutations, which we simply denote by $\leq$. If the permutation π fails to contain σ we say that π *avoids* σ.

A downset in this permutation containment order is referred to as a *permutation class*; in other words, if $\mathcal{C}$ is a permutation class, $\pi \in \mathcal{C}$ and $\sigma \leq \pi$, then $\sigma \in \mathcal{C}$. We denote by $\mathcal{C}_n$ the set $\mathcal{C} \cap S_n$ (the permutations of length n in $\mathcal{C}$) and we refer to $\sum_{n\geq 0} |\mathcal{C}_n| x^n$ as the *generating function of* $\mathcal{C}$. Given any set of permutations B, the set $\mathrm{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$ forms a permutation class and, conversely, for any permutation class $\mathcal{C}$ there is a unique antichain (set of pairwise incomparable elements) B such that $\mathcal{C} = \mathrm{Av}(B)$; we call this antichain the *basis* of $\mathcal{C}$, and say that $\mathcal{C}$ is *finitely based* if B is finite.

One area in which permutation classes arise is the study of sorting machines. For example, Knuth [7] showed the class $\mathrm{Av}(231)$ consists precisely of those permutations that can be sorted by a stack (a last-in first-out linear sorting machine), while Tarjan [12] observed that the class of permutations that can be sorted by a network consisting of two parallel queues (first-in first-out linear sorting machines) is $\mathrm{Av}(321)$. A classic result in the field of permutation containment is that the number of permutations in $\mathrm{Av}(231)$ and in $\mathrm{Av}(321)$ of length n are both equal to the nth Catalan number. For bijections between the two sets, see the recent survey by Claesson and Kitaev [3].

Our interest in this paper is with permutations which "almost lie" in a given permutation class, a concept which we formalize as follows: given a permutation class $\mathcal{C}$ and natural number t, we say that the permutation π *t-almost lies in* (or simply *almost lies in* if $t = 1$) $\mathcal{C}$ if one can remove t (or fewer) entries from π to obtain a permutation that lies in $\mathcal{C}$. We denote the set of permutations which t-almost lie in $\mathcal{C}$ by $\mathcal{C}^{+t}$; note that $\mathcal{C}^{+t}$ is a permutation class itself.

The notion of almost avoiding permutations has a natural interpretation in terms of sorting machines: if $\mathcal{C}$ consists of those permutations which can be sorted by the machine M then the permutations in $\mathcal{C}^{+1}$ are those which can be sorted by M in parallel with a *one-time use buffer*, which we define as a machine which can hold one entry, once in the sorting process. The classes $\mathcal{C}^{+t}$ for $t \geq 2$ then consist of those permutations which can be sorted by M in parallel with t one-time use buffers.

Note that almost avoidance classes differ from the classes introduced by Noonan [9], who studied permutations with at most one copy of 321; this permutation class, which we denote by $\mathrm{Av}(321^{\leq 1})$, is strictly contained in $\mathrm{Av}(321)^{+1}$.

In the following two sections we provide the enumeration of $\mathrm{Av}(231)^{+1}$ and $\mathrm{Av}(321)^{+1}$, i.e., the permutations that can be "almost stack-sorted" and "almost sorted by two parallel queues", and then end with a conjecture. By the usual symmetries of permutations, this completes the enumeration of permutations which almost avoid a pattern of length 3. Before this, we conclude the introduction with a general result, first proved by the Theory of Computing Research Group at the University of Otago in 2002.

Proposition 1 (Otago Theory of Computing Research Group)**.** *For any finitely based class $\mathcal{C}$ and positive integer t, the class $\mathcal{C}^{+t}$ is finitely based.*

Proof. It suffices to prove that $\mathcal{C}^{+1}$ is finitely based if $\mathcal{C}$ is, as then the proposition follows by iteration. Suppose that the longest basis element of $\mathcal{C}$ has length m and consider a permutation $\tau \notin \mathcal{C}^{+1}$. Since $\tau \notin \mathcal{C}$, τ contains a subsequence of length at most m order isomorphic to a basis element of $\mathcal{C}$. Furthermore, since $\tau \notin \mathcal{C}^{+1}$, every time we remove a single element from this subsequence, we find another occurrence of a basis element of $\mathcal{C}$ (which is also of length at most m). By taking the original subsequence together with these additional occurrences of basis elements we see that τ contains a permutation of length at most $m(m+1)$ which does not lie in $\mathcal{C}^{+1}$, verifying that the basis of $\mathcal{C}^{+1}$ is finite. □

2. $\mathbf{Av(321)^{+1}}$

To enumerate the permutations in $\mathrm{Av}(321)^{+1}$ we use the Robinson-Schensted algorithm. While a detailed description of this algorithm can be found in Sagan's text [10], a few details suffice for our arguments. First recall that the Robinson-Schensted algorithm associates to each permutation π of length n a pair, denoted $(P(\pi), Q(\pi))$, of standard Young tableaux (SYT), each with n cells and of the same shape. We denote the shape of $P(\pi)$ by $\mathrm{sh}\ P(\pi)$, so in the case where π is of length n, $\mathrm{sh}\ P(\pi) = \mathrm{sh}\ Q(\pi)$ is a partition, say $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n (which we denote by $\lambda \vdash n$). Schensted [11] proved that the length of the longest decreasing subsequence π is equal to the number of rows of $P(\pi)$, and thus if $\mathrm{sh}\ P(\pi) = \lambda = (\lambda_1, \ldots, \lambda_r)$, then the longest decreasing subsequence of π is of length r. Greene [5] gave a generalization of Schensted's theorem, from which the following proposition routinely follows.

Proposition 2. *If $\mathrm{sh}\ P(\pi) = (\lambda_1, \ldots, \lambda_r)$ then the longest $k \cdots 21$-avoiding subpermutation in π has length $\lambda_1 + \cdots + \lambda_{k-1}$.*

The first step in our enumeration is to characterize the shapes of SYT that can arise from a permutation which almost avoids 321.

Proposition 3. *The permutation π lies in $\mathrm{Av}(321)^{+1}$ if and only if $\mathrm{sh}\ P(\pi)$ is of the form (k), (k, ℓ), or $(k, \ell, 1)$ for some integers $k \geq \ell \geq 1$.*

Proof. No permutation in $\mathrm{Av}(321)^{+1}$ can contain a 4321 pattern as then there would be no entry whose removal gives a 321-avoiding permutation. Similarly, no permutation in $\mathrm{Av}(321)^{+1}$ can contain two disjoint occurrences of 321. This means that such a permutation cannot contain a 123-avoiding permutation of length 6. Translating into SYT, this means that, for all $\pi \in \mathrm{Av}(321)^{+1}$, $\mathrm{sh}\ P(\pi)$ has no columns of length 4 or greater (it avoids 4321)

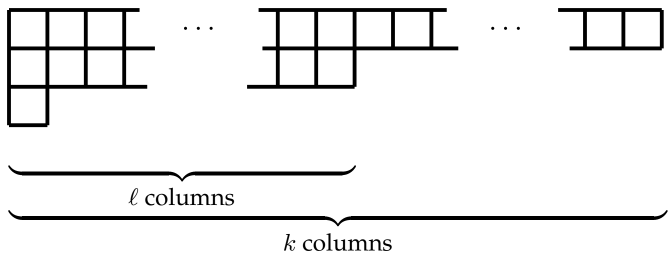

Figure 1: The shape of a Standard Young Tableaux obtained from the Robinson-Schensted algorithm applied to a permutation in $\mathrm{Av}(321)^{+1}$ that contains at least one 321 pattern

and at most one column of length 3 (its longest 123-avoiding subpermutation has length at most 5, so this follows from Proposition 2). This implies that $\mathrm{sh}\ P(\pi)$ is of one of the forms listed in the statement of the proposition. For the other direction, note that if $\mathrm{sh}\ P(\pi) = (k)$ or $\mathrm{sh}\ P(\pi) = (k, \ell)$ then π avoids 321, while if $\mathrm{sh}\ P(\pi) = (k, \ell, 1)$ then π contains a subpermutation of length $k + \ell$ which avoids 321, completing the proof. □

Note that the proof of Proposition 3 shows that $\pi \in \mathrm{Av}(321)^{+1}$ if and only if π avoids 4321 and does not contain two disjoint occurrences of 321. Thus the basis elements of $\mathrm{Av}(321)^{+1}$ are all of length at most 6 and can be readily generated by computer.

By Proposition 3 and the Robinson-Schensted algorithm we now have that

$$|\,\mathrm{Av}_n(321)^{+1}| = \sum_{\lambda=(k,\ell)\vdash n} (f^\lambda)^2 + \sum_{\lambda=(k,\ell,1)\vdash n} (f^\lambda)^2,$$

where f^λ denotes the number of SYT of shape λ. The first sum is simply the number of 321-avoiding permutations which, as stated in the introduction, is equal to the nth Catalan number, C_n. To evaluate the second sum we use the Hook Length Formula, which states that for $\lambda \vdash n$, f^λ is equal to $n!$ divided by the product of the hook lengths of cells in the Ferrers diagram of λ. In the case of $\lambda = (k, \ell, 1)$, the product of the hook lengths of cells in the top row is

$$(k+2)k \cdots (k-\ell+2)(k-\ell) \cdots 1 = \frac{(k+2)k!}{(k-\ell+1)},$$

the product for the middle row is $(\ell+1)(\ell-1)!$, and the solitary cell in the bottom row has a hook length of 1. Thus we have:

$$\begin{aligned} |\,\mathrm{Av}_n(321)^{+1}| &= C_n + \sum_{(k,\ell,1)\vdash n} \left(\frac{n!(k-\ell+1)}{(k+2)k!(\ell+1)(\ell-1)!} \right)^2 \\ &= C_n + \sum_{k=\lfloor n/2 \rfloor}^{n-2} \left(\frac{n!(2k-n+2)}{(k+2)k!(n-k)(n-k-2)!} \right)^2. \end{aligned}$$

An empirical calculation in Maple implies that these numbers likely have the generating function

$$\frac{1-8x+13x^2+24x^3-48x^4-(1-6x+x^2+34x^3-26x^4-4x^5)\sqrt{1-4x}}{2x^2(1-x)(1-4x)^2}.$$

3. $\mathbf{Av(231)^{+1}}$

Our approach to enumerating the class $\mathrm{Av}(231)^{+1}$ differs significantly from the approach used in the previous section and makes use of the following definition: the entry $\pi(i)$ in π is called *essential* if its removal results in a 231-avoiding permutation. For example, the permutation $\pi = 1742653$ contains two essential entries, $\pi(5) = 4$ and $\pi(7) = 3$. (Also note that if $\pi \in \mathrm{Av}(231)$, then by our definition every entry of π is essential.) As a first step, we make the following observation.

Proposition 4. *An essential entry in the permutation $\pi \in \mathrm{Av}(231)^{+1}$ participates as the minimum entry in either all or none of the occurrences of 231 in π.*

Proof. Suppose, to the contrary, that $\pi \in \mathrm{Av}(231)^{+1}$ contains indices $i < j < k$ such $\pi(i)\pi(j)\pi(k)$ is order isomorphic to 231 and $\pi(k)$ is essential, but that $\pi(k)$ also participates in another 231 pattern as a non-minimal element. Label the minimal element of this later pattern $\pi(\ell)$. Clearly $\pi(i)\pi(j)\pi(\ell)$ is also order isomorphic to 231, so $\pi - \pi(k) \notin \mathrm{Av}(231)$, a contradiction to the assumption that $\pi(k)$ is essential. □

By Proposition 4 we can divide the essential entries of a permutation $\pi \in \mathrm{Av}(231)^{+1}$ into *small essential entries*, which participate as the minimum entry in all occurrences of 231, and *large essential entries*, which participate as the minimum entry in no occurrences of 231.

In enumerating $\mathrm{Av}(231)^{+1}$, we use two generating functions frequently,

$$\begin{aligned} f &= \text{the generating function for } \mathrm{Av}(231)^{+1}\text{, and} \\ c &= \text{the generating function for the Catalan numbers, so also for } \mathrm{Av}(231), \\ &= \frac{1-\sqrt{1-4x}}{2x}. \end{aligned}$$

Both of these generating functions have constant term 1.

Proposition 5. *The generating function for permutations in $\mathrm{Av}(231)^{+1} \setminus \mathrm{Av}(231)$ in which the greatest element is not involved in a copy of 231 is given by $2(f-c)xc$.*

Proof. The plot of a permutation of the specified form can be divided into the greatest entry and two regions, A and B, as depicted in Figure 2. One (but not both) of the two regions A or B must contain a permutation in $\mathrm{Av}(231)^{+1} \setminus \mathrm{Av}(231)$, while the other must contain a 231-avoiding permutation. This leads to the generating function specified in the proposition. □

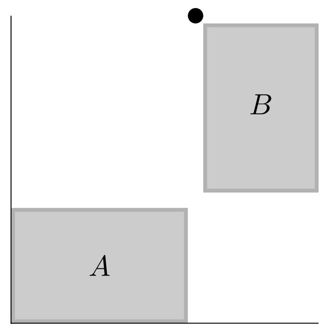

Figure 2: A permutation in $\mathrm{Av}(231)^{+1} \setminus \mathrm{Av}(231)$ in which the greatest element is not involved in a copy of 231.

Proposition 6. *The generating function for permutations in* $\mathrm{Av}(231)^{+1}$ *with an essential greatest (resp., leftmost, rightmost, or least) entry is* $x^2c' + xc - c + 1$.

Proof. The cases are all similar, so we count permutations with an essential greatest entry. To construct such a permutation, one must insert a new greatest entry into a 231-avoider (such permutations have the generating function $x^2c' + xc$ because there are $n+1$ ways to insert a new greatest entry into any 231-avoider, and the number of 231-avoiders is the nth Catalan number) without creating a 231-avoider (these have the generating function $c - 1$). ☐

For the following results we need a bit of notation; for a permutation $\pi \in S_n$ and sets $A, B \subseteq [n]$, we write $\pi(A \times B)$ for the permutation which is order isomorphic to the subsequence of π which has indices from A and values in B.

Proposition 7. *The generating function for permutations in* $\mathrm{Av}(231)^{+1}$ *with an essential small entry in which the greatest entry participates in a copy of* 231 *but is not essential is* $x^4c'^2 + xc(x^2c' + xc - c + 1)$.

Proof. Take π of length n specifying the hypotheses, and suppose that $\pi(j) = n$ and that the small essential entry is $\pi(k)$. As n must be involved in at least one copy of 231, $\pi([1, j) \times (\pi(k), n))$ must be nonempty; let $\pi(i)$ denote the greatest entry in this region. By this choice of i, $\pi([1, j) \times (\pi(i), n))$ is empty, and because $\pi(k)$ is essential, $\pi((j, n] \times [1, \pi(i)))$ contains only the entry $\pi(k)$.

We therefore have three types of entries: the small essential $\pi(k)$, the entries in $\pi([1, j) \times [1, \pi(i)])$, and the entries in $\pi((j, n] \times (\pi(i), n]))$. We further divide these latter regions into A, B, C and D, as indicated in Figure 3.

As the entry n is not essential, one of two situations must occur:

(S1) π has an entry in C, or

(S2) $\pi(k)$ forms a copy of 231 with two entries from B.

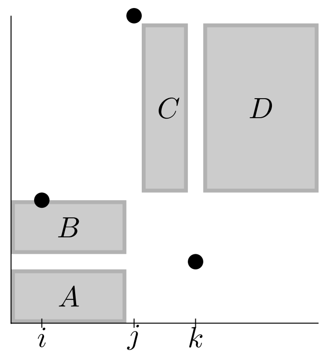

Figure 3: A permutation with a small essential entry in which the greatest entry is involved in at least one copy of 231 but is not essential.

Conversely, any permutation of this form that satisfies (S1) or (S2) is of the desired form.

First we count the permutations satisfying (S1). In this case the entries in $A \cup B$ form a 231-avoiding permutation, as do the entries in $C \cup D$; thus both sets of entries are counted by c. The generating function for arrangements of $\pi(k)$ among the entries in $A \cup B$ is therefore x^2c', and this is the same as the generating function for arrangements of $\pi(k)$ among the entries in $C \cup D$. Multiplying these functions together double-counts the entry $\pi(k)$, but fails to count $\pi(n)$, so the total contribution of the permutations of this type satisfying (S1) is $x^4c'^2$.

Now we need to count the permutations that satisfy (S2) but not (S1). We know that π does not have a entry in C and that the entries of π in D avoid 231. Furthermore, $\pi(k)$ and the entries in $A \cup B$ form a permutation with an essential rightmost entry, so their generating function is $x^2c' + xc - c + 1$ by Proposition 6. After taking into account the contribution of n, permutations of this type contribute $xc(x^2c' + xc - c + 1)$, proving the proposition. □

Proposition 8. *The generating function for permutations in* $\mathrm{Av}(231)^{+1}$ *without a small essential entry, with a large essential entry that is not the greatest entry, and in which the greatest entry participates in at least one copy of* 231 *is given by* $x^2c'(x^2c' + xc - c + 1) + (x^2c' + x^2c - c + x + 1)(c - 1)$.

Proof. By arguments analogous to the proof of Proposition 7 it can be established that these permutations are of the form depicted in Figure 4 (in this figure the essential large entry is $\pi(i)$). Note that, as indicated by the figure, π must contain a entry in region C as otherwise the greatest element of π would not lie in a copy of 231. We divide these permutations into two types:

(L1) there is a copy of 231 containing $\pi(i)$ and entries in $A \cup B$ or

(L2) there is no such copy of 231.

In case (L1), the entries in $A \cup B$ must avoid 231 because $\pi(i)$ is essential, so they are counted by c. Thus x^2c' is the generating function for the number of arrangements of $\pi(i)$

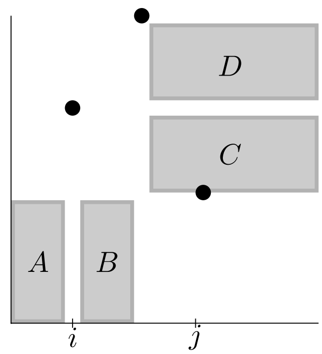

Figure 4: A permutation with a large essential entry in which the greatest entry participates in a copy of 231 but is not essential.

together with the entries in $A \cup B$. We then have that $\pi(i)$ is a leftmost essential element of the permutation given by it and the entries in $C \cup D$, so Proposition 6 shows that these entries are counted by $x^2c' + xc - c + 1$. Multiplying these functions double-counts $\pi(i)$ but does not count the greatest entry of π, so the contribution of the (L1) permutations is $x^2c'(x^2c' + xc - c + 1)$.

In case (L2), $\pi(i)$ and the entries in $A \cup B$ form a nonempty 231-avoiding permutation, and are thus counted by $c - 1$. The entries in $C \cup D$ together with $\pi(i)$ form a permutation with an essential leftmost entry, but we cannot directly apply Proposition 6 because if π had a unique entry in C then that entry would be a small essential entry, and we do not wish to count such permutations. Thus we subtract the generating function for permutations with an essential leftmost entry of value 2; it is easily seen that this generating function is $x(c - xc - 1)$, and so the contribution of permutations in case (L2) is $(x^2c' + x^2c - c + x + 1)(c - 1)$, completing the proof. □

Theorem 9. *The generating function for the permutations that almost avoid* 231 *is*

$$\frac{1 - 5x - 6x^2 + 45x^3 - 24x^4 - (1 + x - 4x^2 + x^3)(1 - 4x)^{3/2}}{-2x^2(1 - 4x)^{3/2}}.$$

Proof. Letting f denote the generating function for permutations that almost avoid 231, Propositions 5–8 show that

$$\begin{aligned} f = c + 2xc(f - c) + x^2c' + xc - c + 1 + x^4c'^2 + xc(x^2c' + xc - c + 1) \\ + x^2c'(x^2c' + xc - c + 1) + (x^2c' + x^2c - c + x + 1)(c - 1), \end{aligned}$$

from which the desired solution follows. □

4. OPEN PROBLEMS

Our computations indicate that $|\operatorname{Av}_n(321)^{+1}| < |\operatorname{Av}_n(231)^{+1}|$ for all $n \geq 4$, which begs for a combinatorial explanation:

Problem 10. *Construct a length-preserving injection from* $\mathrm{Av}(321)^{+1}$ *to* $\mathrm{Av}(231)^{+1}$.

Finally, we conclude with a conjecture about the exact enumeration problem.

Conjecture 11. *For all* t*, the generating functions for* $\mathrm{Av}(231)^{+t}$ *and* $\mathrm{Av}(321)^{+t}$ *are rational in* x *and* $\sqrt{1-4x}$.

We note that there has been similar work done for sets of permutations with at most t copies of these patterns. Bóna [1] proved that the generating function for $\mathrm{Av}(231^{\leq t})$ is rational in x and $\sqrt{1-4x}$ (see also Mansour and Vainshtein [8] and Brignall, Huczynska, and Vatter [2]). For the pattern 321, Noonan [9] enumerated $\mathrm{Av}(321^{\leq 1})$, while Fulmek [4] counted $\mathrm{Av}(321^{\leq 2})$ and conjectured that the generating function for $\mathrm{Av}(321^{\leq t})$ is rational in x and $\sqrt{1-4x}$.

The study of almost avoidance is extended to pairs of permutations of length 3 by Griffiths, Smith, and Warren [6].

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