# Notes on maximal Frobenius numbers

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ABSTRACT. Fixing a positive integer $L$, we answer the question "How large can the Frobenius number of a set of divisors of $L$ get?"

In §2 we describe an open question, coming from brick tilings, involving Frobenius numbers. §3 gives a Frobenius result for a *pair* of numbers. This result is due to Dubins & Meilijson.

[Acknowledgments: Benjamin.J.Tilly@Dartmouth.EDU sent me arguments about the Frobenius number. I've jazzed-up his arguments into Prop. B below.

My proof dates from some time in Feb1996.]

The Frobenius number gives information as to which intervals can be packed by a fixed set of intervals. A multi-dimensional version of the Frobenius number arises in [Kin1]. It leads to a question which is posed at the end of this note. A weaker form of the question is answered by Proposition B, below.

In the sequel, "$a \perp b$" means that $a$ and $b$ are co-prime, and the prefix ***nv-*** abbreviates *non-void*, e.g, "a nv-subset". and $G \coloneqq \gcd(\mathcal{B})$.

## §1 PRELIMINARIES

Say that a multiple of $G$, call it $n$, is ***good*** (more precisely, $\mathcal{B}$-***good***) if it can be expressed as a non-negative linear combination (NNLC) of the members of $\mathcal{B}$,

$$n = \sum_{b \in \mathcal{B}} \gamma_b \cdot b\,, \qquad \text{each coefficient } \gamma_b \in \mathbb{N}$$

—if not, then $n$ is ***bad***. Thus the bad numbers union the good numbers equals $G\mathbb{Z}$. The ***Frobenius number***, written $\mathrm{Frob}(\mathcal{B})$, is the most-positive[¥] bad number. The following is straightforward to establish.

[¥] $\mathrm{Frob}(\mathcal{B})$ is positive –at least as large as $G$ is– unless $G \in \mathcal{B}$, in which case $\mathrm{Frob}(\mathcal{B}) = -G$.

FACT 1. *If $a \perp b$, then*

$$\mathrm{Frob}\{a, b\} = (a-1)(b-1) - 1 = ab - (a+b)\,.$$

*More generally,*

$$\mathrm{Frob}\,\{a, b\} = \tfrac{ab}{g} - (a+b)\,,$$

*where $g \coloneqq \gcd\{a, b\}$.*

As an example, $\mathrm{Frob}\{10, 14\} = 46$; so 46 is not a NNLC of 10 and 14, yet every higher even number is.

Now (1) shows that the Frobenius number of a pair is alway dominated by its lcm. However, this may fail for triples.

**Example.** The lcm of $\{10, 14, 35\}$ is 70, and their gcd is 1. Even though 81 exceeds 70, and is a multiple of 1, it fails to be a NNLC of $\{10, 14, 35\}$.

Were it so, the NNLC would have to use all three numbers. But 35 can not be used twice or more, so $81 - 35$ would have to be a NNLC of $\{10, 14\}$. Yet $81 - 35$ equals 46. ♦

**Notation.** Let $\coloneqq$ mean "is defined to be" and let $\equiv_L$ mean "congruent modulo $L$". Use $a \triangleleft b$ for "$a$ divides $b$" and use $b \triangleright a$ for "$b$ is a multiple of $a$".

I employ $[a\,..\,b)$ for the half-open "interval of integers" $[a, b) \cap \mathbb{Z}$, and use $\mathbb{N} = [0\,..\,\infty)$ for the natural numbers. For any number $b$, let $\mathcal{B}_{\neq b}$ mean $\mathcal{B} \smallsetminus \{b\}$. Let $\#\mathcal{B}$ indicate the cardinality of $\mathcal{B}$.

Always, $L$ is a positive integer and Pr is the set of primes in the factorization of $L$, where

$$L = \prod_{p \in \mathrm{Pr}} p^{\ell[p]}\,, \qquad \text{with } \ell[p] \in \mathbb{Z}_+. \tag{2}$$

Let $\mathrm{Log}_p(b)$ mean the exponent of $p$ in the prime factorization of $b$; so $\mathrm{Log}_p(L)$ equals $\ell[p]$.

DEFINITION. A family, $\mathcal{B}$, with at least two members, is ***tempered*** if: The number $L \coloneqq \mathrm{lcm}(\mathcal{B})$ is *not* in $\mathcal{B}$ and

$$\forall b \in \mathcal{B}\colon\;\; \mathrm{lcm}\big\{\gcd(\mathcal{B}_{\neq b}),\, b\big\} \;=\; L.$$

Necessarily, such an $L$ is composite.

Theorem A. *For every finite, nv-family $\mathcal{B}$:*

$$(3i)\qquad \operatorname{Frob}(\mathcal{B}) \;\le\; \big[\#\mathcal{B}-1\big]\operatorname{lcm}(\mathcal{B}) - \sum \mathcal{B}\,.$$

*If the family is tempered, or is a singleton, then there is equality:*

$$(3ii)\qquad \operatorname{Frob}(\mathcal{B}) \;=\; \big[\#\mathcal{B}-1\big]\operatorname{lcm}(\mathcal{B}) - \sum \mathcal{B}\,.$$

Proof of 3*i*. Let $L \coloneqq \operatorname{lcm}(\mathcal{B})$. Take a number $n \in \gcd(\mathcal{B})\mathbb{Z}$ such that

$$(4)\qquad n \;>\; \big[\#\mathcal{B}-1\big]L - \sum \mathcal{B}\,.$$

It suffices to show that $n$ is good.

Take integers $\{\gamma_b\}_{b\in\mathcal{B}}$ so that $n = \sum_{b\in\mathcal{B}} \gamma_b \cdot b$. For each $b$ in $\mathcal{B}$, let $L_b$ denote $L/b$ and let $\widehat{\gamma}_b$ be the residue of $\gamma_b$ modulo $L_b$; so $\widehat{\gamma}_b \in [0\,..\,L_b)$. Thus

$$(5)\qquad \widehat{\gamma}_b \cdot b \;\le\; (L_b - 1)b \;\stackrel{\text{note}}{=\!=}\; L - b\,.$$

Further, since $L_b \cdot b$ equals $L$,

$$[\gamma_b - \widehat{\gamma}_b]\cdot b \;\equiv_L\; 0\,.$$

Consequently $\widehat{\gamma}_b \cdot b \equiv_L \gamma_b \cdot b$, and so

$$\widehat{n} \;\coloneqq\; \sum_{b\in\mathcal{B}} \widehat{\gamma}_b b \;\equiv_L\; \sum_{b\in\mathcal{B}} \gamma_b b \;=\; n\,.$$

By (5), this $\widehat{n} \le \#\mathcal{B}\cdot L - \sum\mathcal{B}$, so (4) implies that $n - \widehat{n} > -L$. Since $n \equiv_L \widehat{n}$, then, $n \in (L\mathbb{N}) + \widehat{n}$. This, together with the goodness of $L$ and $\widehat{n}$, implies that $n$ is good. ♦

Proof of 3*ii*. Since equality (3*ii*) is invariant under division by $\gcd(\mathcal{B})$, without loss of generality $\gcd(\mathcal{B}) = 1$. Thus $L_b \perp b$, where

$$L_b \;\coloneqq\; \gcd(\mathcal{B}_{\neq b})\,.$$

Since $\mathcal{B}$ is tempered, $L_b \cdot b = L$. As our goal is to show that $\operatorname{Frob}(\mathcal{B})$ dominates $\big[\#\mathcal{B}-1\big]L - \sum\mathcal{B}$, we need but show that this latter quantity is bad.

*The Estimate.* Now consider a good number of the form $xL - \sum\mathcal{B}$, with $x$ an integer. By definition there exist non-negative coefficients $\gamma_b$ so that

$$(6)\qquad xL - \sum_{b\in\mathcal{B}} b \;=\; \sum_{b\in\mathcal{B}} \gamma_b \cdot b\,.$$

Fix a particular $b$ and reduce this equality modulo $L_b$ to conclude

$$x\cdot 0 - b \;\equiv_{L_b}\; \gamma_b \cdot b\,.$$

Thus $L_b$ divides $(1+\gamma_b)b$, hence divides $(1+\gamma_b)$, since $L_b \perp b$. Therefore $1+\gamma_b \ge L_b$, since $1+\gamma_b$ is positive. Multiplying by $b$ gives

$$\gamma_b \cdot b \ge L - b\,.$$

We may conclude, using (6), that

$$xL - \sum\mathcal{B} \;\ge\; \sum_{b\in\mathcal{B}} (L-b) \;=\; \#\mathcal{B}\cdot L - \sum\mathcal{B}\,.$$

Thus $x \ge \#\mathcal{B}$.

The upshot is that $[\#\mathcal{B}-1]L - \sum\mathcal{B}$ is *not* good. And since $\gcd(\mathcal{B})$ divides this quantity, it is bad. ♦

**Tempered description.** Fix a composite $L$, factored as (2). Each non-negative integer-valued function $n[\cdot]$ which is *strictly* exceeded by $\ell[\cdot]$, determines a tempered family whose lcm is $L$. The family is

$$\mathcal{C}^{\langle n\rangle} \;\coloneqq\; \left\{ \frac{L}{p^{\ell[p]-n[p]}} \;\middle|\; p \in \Pr \right\}.$$

The following observation is immediate.

Exercise 7. *Suppose $\mathcal{B}$ is tempered and $\operatorname{lcm}(\mathcal{B})$ equals $L$. Then,*

$$\#\mathcal{B} \;\le\; \#\Pr\,.$$

*There is equality iff $\mathcal{B}$ is one of the $\mathcal{C}^{\langle\rangle}$ families.*

Definition. Let $\mathrm{maxFr}(L)$ be the maximum of $\mathrm{Frob}(\mathcal{B})$ as $\mathcal{B}$ ranges over all non-empty subsets of $\mathrm{Divisors}(L)$, the set of divisors of $L$. The uninteresting case is when $L$ is a power of a prime; then $\mathrm{maxFr}(L)$ is $-L$. The next result handles composite $L$.

Theorem B. *When $L$ is composite, then*

$$\text{(8)} \qquad \mathrm{maxFr}(L) = \Big(\#\mathrm{Pr} - 1 - \sum_{p\in\mathrm{Pr}} \frac{1}{p^{\ell[p]}}\Big) \cdot L\,.$$

Proof of B. Since $L$ is composite, $\mathrm{maxFr}(L)$ is positive. Thus throughout the replacements described below, $\mathcal{B}$ always has at least two members. (Frob of a singleton is negative.) So for each prime in Pr, at least one replacement will occur. Thus at the end of the process, $\mathrm{lcm}(\mathcal{B})$ will equal $L$.

Take a nv-collection $\mathcal{B} \subset \mathrm{Divisors}(L)$ maximizing $\mathrm{Frob}(\mathcal{B})$. Let $m[p]$ be the minimum, as $b$ varies over $\mathcal{B}$, of $\mathrm{Log}_p(b)$. Thus

$$\gcd(\mathcal{B}) = \prod_{p\in\mathrm{Pr}} p^{m[p]}\,.$$

Fix a prime $p \in \mathrm{Pr}$, then fix a member $b \in \mathcal{B}$ with $\mathrm{Log}_p(b) = m[p]$. Now, for each other member $d \in \mathcal{B}_{\neq b}$, replace $d$ in $\mathcal{B}$ by

$$d' := \mathrm{lcm}\{d, p^{\ell[p]}\}\,,$$

which is a divisor of $L$. Since $d \trianglelefteq d'$ and $\gcd(\mathcal{B})$ is unchanged, the replacement can not have decreased $\mathrm{Frob}(\mathcal{B})$. Observe that this operation *can* make distinct members of $\mathcal{B}$ equal. Thus it might reduce the cardinality of $\mathcal{B}$.

Having done this operation for each prime $p$, now $\mathrm{lcm}(\mathcal{B}) = L$. Further, since family $\mathcal{B}$ has at least two members, the family can be replaced by $\mathcal{B}_{\neq L}$ and not change its Frobenius number. Consequently, $\mathcal{B}$ is tempered.

*Maximizing* $\mathrm{Frob}(\mathcal{B})$. Courtesy (3*ii*), the Frobenius number equals $[\#\mathcal{B} - 1]L - \sum \mathcal{B}$. A maximum of the latter can occur only if $\#\mathcal{B}$ is maximized, since $L$ exceeds each member of $\mathcal{B}$. Courtesy (7), then, $\mathcal{B}$ must be one of the $\mathcal{C}^{\langle\rangle}$ families. So

$$\mathrm{maxFr}(L) = \mathrm{Frob}\big(\mathcal{C}^{\langle n\rangle}\big) = [\#\mathrm{Pr} - 1]L - \sum \mathcal{C}^{\langle n\rangle}\,,$$

where the $n[]$ function is yet to be determined.

The last step is to take $n[]$ so as to minimize the sum of the numbers in $\mathcal{C}^{\langle n\rangle}$. Evidently, this occurs (uniquely) for the identically-zero function. Since

$$\text{(8}'\text{)} \qquad \mathcal{C}^{\langle 0\rangle} = \Big\{\frac{L}{p^{\ell[p]}} \;\Big|\; p \in \mathrm{Pr}\Big\}\,,$$

we obtain (8), as desired. ♦

Corollary B′. *When a subfamily of* $\mathrm{Divisors}(L)$ *has maximum Frobenius number then its* lcm *must be $L$. If, in addition, it is tempered, then it must be the* (8′) *family.*

## §2 Question from Brick-packing

Given $N$ and an upper bound $U$, let $f(N, U)$ be the maximum value of $\mathrm{Frob}(\mathcal{B})$ over all collections $\mathcal{B}$ obtained as follows:

> *Pick an arbitrary set $\mathcal{A}$ of $N$ positive integers, each less-equal $U$. Pick an arbitrary nv-collection $\mathcal{B}$, where each $b \in \mathcal{B}$ is the least common multiple of some subset of $\mathcal{A}$.*

*How fast does $f(N, U)$ grow?*

## §3 The Sporadic Numbers

Given a collection $\mathcal{B}$ and letting $F$ be $\text{Frob}(\mathcal{B})$, I call those $\mathcal{B}$-good numbers in the interval $[0 \mathinner{..} F]$, the ***sporadic*** numbers. In the case that $\mathcal{B}$ is a co-prime pair $\{J, K\}$, the sporadic numbers can be characterized by (9), below.

Each integer $r$ can be expressed as $Jr + Ks$, with $r$ and $s$ integers. If $r$ is restricted in a range of some $K$ consecutive integers, then the pair $r, s$ is unique. Let $\widehat{r}$ and $\widetilde{r}$ denote the unique pair of integers for which

$$r \;=\; J\widehat{r} + K\widetilde{r}\,, \quad \text{with } \widehat{r} \in [0 \mathinner{..} K).$$

Evidently, for each integer $r$,

$$r \textit{ is good} \;\text{ IFF }\; \widetilde{r} \geq 0\;. \tag{9}$$

Theorem 10 [Lester Dubins & Isaco Meilijson]. *Suppose $J \perp K$ and let $F \coloneqq \text{Frob}\{J, K\}$. Then the involution (of $\mathbb{Z}$)*

$$\varphi \;\coloneqq\; \langle r \mapsto F - r \rangle$$

*maps bad numbers to good, and vice versa. Consequently, since $\varphi$ maps $[0 \mathinner{..} F]$ to itself, exactly half the numbers in $[0 \mathinner{..} F]$ are good.* (Note that $F$ is necessarily odd, since $J \perp K$.)

Proof. Let $s \coloneqq F - r$ and observe that

$$J \cdot \left[\widehat{r} + \widehat{s}\right] \;+\; K \cdot \left[\widetilde{r} + \widetilde{s}\right] \;=\; r + s$$

$$=\; F \;=\; J\widehat{F} \;+\; K\widetilde{F}\,.$$

Hence there is congruence modulo $K$,

$$\widehat{r} + \widehat{s} \;\equiv_K\; \widehat{F}\,. \tag{11}$$

Now $\widehat{F} = K - 1$, by Fact 1. And $\widehat{r}, \widehat{s} \in [0 \mathinner{..} K)$, so $0 \leq \widehat{r} + \widehat{s} < \widehat{F} + K$. Consequently, we must have actual equality in (11). Hence

$$\widetilde{r} + \widetilde{s} \;=\; \widetilde{F}$$

$$=\; -1\,. \qquad \text{(By Fact 1)}$$

So exactly one of $\widetilde{r}$ and $\widetilde{s}$ is non-negative. ♦

## References

[Kin1] J.L. King, *Brick Tiling and Monotone Boolean Functions*, Preprint available at: `http://www.math.ufl.edu/~squash/tilingstuff.html`.

[D&M] L. Dubins & I. Meilijson, Preprint.