A (More Detailed) Introduction to Constructive Algebraic Quantum Field Theory

Unless stated otherwise, $d = 4$

Crucial to all purely algebraic constructions so far have been algebras associated with wedge regions (*e.g.* $\mathcal{W}_R \doteq \{x \in \mathcal{M}^4 \mid x_1 > |x_0|\}$).

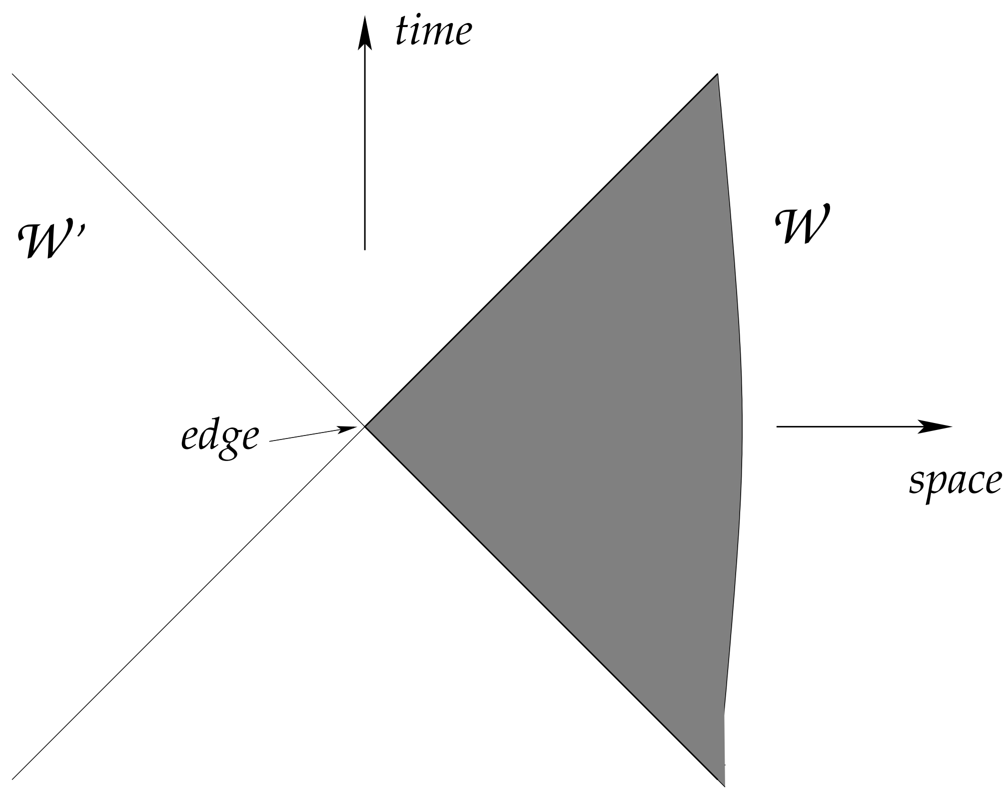

Figure 1: A wedge $\mathcal{W}$, its causal complement $\mathcal{W}'$ and their common edge

$\boldsymbol{\mathcal{W}} = \{\lambda \mathcal{W} \mid \lambda \in \mathcal{P}_+^\uparrow\} \quad (d > 2)$

Basic Observation

For any (closed) convex, causally complete ($\mathcal{O} = \mathcal{O}''$),

$$\mathcal{O} = \bigcap \{\overline{\mathcal{W}} \mid \mathcal{O} \subset \mathcal{W} \, , \, \mathcal{W} \in \boldsymbol{\mathcal{W}}\} \, .$$

(Thomas & Wichmann, 1997)

Examples: double cones, spacelike cones, spacelike cylinders ($\mathcal{O} = \mathcal{W}_1 \cap \mathcal{W}_2{}'$ for $\overline{\mathcal{W}_2} \subset \mathcal{W}_1$).

Purely Algebraic Construction Strategies

- (Wiesbrock): Construct a small number of von Neumann algebras satisfying certain relations (involving their modular structure with respect to a certain state vector) and from them to generate the net $\{\mathcal{A}(\mathcal{O})\}$ (in which the initial algebras are re-interpreted as wedge algebras).
- (Buchholz & Lechner, Buchholz & S., Longo & Rehren) Construct a net of nonlocal wedge algebras and use suitable relative commutants of such algebras to construct a net $\{\mathcal{A}(\mathcal{O})\}$ of local algebras.

- (Buchholz & S.): Construct a representation $U(\mathcal{P}_+^\uparrow)$ satisfying the spectrum condition (it would suffice to do so on Fock space) and for a fixed wedge $\mathcal{W}_0$ exhibit an algebra $\mathfrak{G}$ which satisfies the consistency conditions:

  (a) $U(\lambda)\mathfrak{G}U(\lambda)^{-1} \subset \mathfrak{G}$ whenever $\lambda\mathcal{W}_0 \subset \mathcal{W}_0$ for $\lambda \in \mathcal{P}_+^\uparrow$.

  (b) $U(\lambda')\mathfrak{G}U(\lambda')^{-1} \subset \mathfrak{G}'$ whenever $\lambda' \mathcal{W}_0 \subset \mathcal{W}_0'$ for $\lambda' \in \mathcal{P}_+^\uparrow$.

The representation $U(\mathcal{P}_+^\uparrow)$ would be fixed by an analysis of particle masses, types and multiplicities in scattering experiments.

The nontrivial part of the problem is finding the algebra $\mathfrak{G}$ satisfying the consistency conditions.

Any algebra $\mathfrak{G}$ satisfying these conditions is the germ of a quantum field theory in the following sense: setting

$$\mathcal{A}(\mathcal{W}) \doteq U(\lambda)\mathfrak{G}U(\lambda)^{-1} \,,$$

where $\lambda \in \mathcal{P}_+^\uparrow$ satisfies $\mathcal{W} = \lambda\mathcal{W}_0$ for given $\mathcal{W} \in \boldsymbol{\mathcal{W}}$, then the definition of the wedge algebras $\mathcal{A}(\mathcal{W})$ is consistent and satisfies the conditions of isotony, covariance and locality.

Well Defined:

$\lambda_1\mathcal{W}_0 = \mathcal{W} = \lambda_2\mathcal{W}_0$ entails $\lambda_2^{-1}\lambda_1\mathcal{W}_0 = \mathcal{W}_0$. So by consistency condition (a)

$$\begin{aligned} U(\lambda_1)\mathfrak{G}U(\lambda_1)^{-1} &= U(\lambda_2)U(\lambda_2^{-1}\lambda_1)\mathfrak{G}U(\lambda_2^{-1}\lambda_1)^{-1}U(\lambda_2)^{-1} \\ &\subset U(\lambda_2)\mathfrak{G}U(\lambda_2)^{-1} \,. \end{aligned}$$

Interchange 1 and 2 to get $U(\lambda_2)\mathfrak{G}U(\lambda_2)^{-1} \subset U(\lambda_1)\mathfrak{G}U(\lambda_1)^{-1}$.

Covariance and Isotony

Covariance: Note that $\mathcal{A}(\mathcal{W}_0) = \mathfrak{G}$, since $\iota\mathcal{W}_0 = \mathcal{W}_0$ and $U(\iota) = \mathbf{1}$. Definition yields for $\mathcal{W} = \lambda_0\mathcal{W}_0$ and any $\lambda \in \mathcal{P}_+^\uparrow$

$$\begin{aligned} U(\lambda)\mathcal{A}(\mathcal{W})U(\lambda)^{-1} &= U(\lambda)U(\lambda_0)\mathcal{A}(\mathcal{W}_0)U(\lambda_0)^{-1}U(\lambda)^{-1} \\ &= U(\lambda\lambda_0)\mathcal{A}(\mathcal{W}_0)U(\lambda\lambda_0)^{-1} \\ &= \mathcal{A}(\lambda\lambda_0\mathcal{W}_0) = \mathcal{A}(\lambda\mathcal{W}) \, . \end{aligned}$$

Isotony: Note that $\mathcal{W}_1 \subset \mathcal{W}_0$ if and only if $\exists x \in \mathcal{W}_0$ such that $\mathcal{W}_1 = \mathcal{W}_0 + x$. So

$$\mathcal{A}(\mathcal{W}_1) = U(x)\mathfrak{G}U(x)^{-1} \subset \mathfrak{G} = \mathcal{A}(\mathcal{W}_0) \, ,$$

by consistency condition (a). Isotony for the wedge net then follows from covariance.

Locality

Locality: Note that $\mathcal{W}_0' = D\mathcal{W}_0$, for a suitable rotation D (*e.g.* if $\mathcal{W}_0 = \mathcal{W}_R$, then one choice is $D = R_2(\pi)$), so

$$\mathcal{A}(\mathcal{W}_0') = U(R)\mathfrak{G}U(R)^{-1} \subset \mathfrak{G}' = \mathcal{A}(\mathcal{W}_0)' \, ,$$

by consistency condition (b). Locality for the wedge net then follows from covariance and isotony.

The algebras corresponding to arbitrary causally closed convex regions can then consistently be defined by taking intersections of wedge algebras:

$$\mathcal{A}(\mathcal{O}) \doteq \bigcap_{\mathcal{W} \supset \mathcal{O}} \mathcal{A}(\mathcal{W}) \,,$$

yielding a covariant and local net.

Viewed in these terms, Wiesbrock (Kähler & Wiesbrock, 2001) gives sufficient conditions to generate $(U(\mathcal{P}_+^\uparrow), \mathfrak{G})$ satisfying the consistency conditions.

Conversely, any asymptotically complete quantum field theory with the given particle content fixes an algebra $\mathfrak{G}$ (namely, the observable algebra associated with $\mathcal{W}_0$) with the above properties. Thus any decent quantum field theory can, in principle, be presented in this way. However, at present a dynamical principle by which the algebras $\mathfrak{G}$ can be selected is missing.

A (Unsuccessful) Dynamical Principle

Baumgärtel and Wollenberg (1984): Given a massive free scalar field ϕ on Fock space $\mathcal{H}$, an associated representation $U(\mathcal{P}_+^\uparrow)$ and a unitary S on $\mathcal{H}$ which is CPT– and Poincaré invariant, there exists a weakly local but nonlocal field φ covariant under $U(\mathcal{P}_+^\uparrow)$ whose associated scattering matrix is S in the sense

$$\lim_{t \to \mp\infty} \langle \Phi, e^{itH} \varphi(f_t) e^{-itH} \Psi \rangle = \begin{matrix} \langle \Phi, \phi(f)\Psi \rangle & , & t \to -\infty \\ \langle S\Phi, \phi(f) S\Psi \rangle & , & t \to \infty \end{matrix}$$

where $\widetilde{f}_t(\vec{p}, p_0) = \exp(-it(\vec{p}^2 + m^2)^{1/2}) \widetilde{f}(\vec{p}, p_0)$. In fact, $\varphi(0) \doteq V\phi(0)V^*$, for suitable choice of unitary V. They remark that they could not prove that the vacuum is cyclic for φ. And, unfortunately, it is not cyclic for the wedge algebras generated by φ. (Buchholz and S.)

Concrete Construction Techniques

Background and Motivation for Modular Localization

Given: von Neumann algebra $\mathcal{M}$ with a cyclic and separating vector Ω. With

$$S A\Omega \doteq A^*\Omega \, , \; A \in \mathcal{M} \, ,$$

the polar decomposition $S = J\Delta^{1/2}$ yields a unique antiunitary involution J and positive (unbounded) Δ such that $J\Omega = \Omega = \Delta\Omega$,

$$J\mathcal{M}J = \mathcal{M}' \quad , \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$$

for all $t \in \mathbb{R}$ (Tomita, 1967; Takesaki, 1970).

In typical vacuum representations the vacuum vector is cyclic and separating for wedge algebras (Reeh–Schlieder, 1961, and beyond). So

$$(\mathcal{A}(\mathcal{W}), \Omega) \mapsto (J_{\mathcal{W}}, \Delta_{\mathcal{W}})$$

Theorem 1 (Bisognano & Wichmann, 1975)**.** *Given a net of von Neumann algebras* $\{\mathcal{A}(\mathcal{O})\}$ *locally associated with a quantum field satisfying the Wightman axioms (hence in a vacuum representation), one has*

$$J_{\mathcal{W}_R} = \Theta U(R_1(\pi)) \doteq U(\theta_R) \, , \, \Delta_{\mathcal{W}_R}^{it} = U(v_R(2\pi t))$$

where Θ *is the PCT-operator associated to the Wightman field and* $R_1(\pi)$ *is the rotation through the angle* π *about the* 1*-axis. Hence,*

$$J_{\mathcal{W}_R}\mathcal{A}(\mathcal{O})J_{\mathcal{W}_R} = \mathcal{A}(\theta_R\mathcal{O}) \, , \, \Delta_{\mathcal{W}_R}^{it}\mathcal{A}(\mathcal{O})\Delta_{\mathcal{W}_R}^{-it} = \mathcal{A}(v_R(2\pi t)\mathcal{O}) \, ,$$

for all $\mathcal{O}$*.*

Note: $\Delta_{\mathcal{W}_R}^{it} = U(v_R(2\pi t)) = e^{i2\pi t K_1}$ entails $\Delta_{\mathcal{W}_R}^{1/2} = e^{\pi K_1}$.

Modular Localization

Brunetti, Guido & Longo (2002) , Schroer (1997 . . .) ($d \geq 2$)

Initial data: strongly continuous (anti)unitary representation $U_1(\mathcal{P}_+)$ on $\mathcal{H}_1$.

Yields $U(\mathcal{P}_+)$ on $\mathcal{H}$, bosonic Fock space with one–particle space $\mathcal{H}_1$.

Let $U_1(v_R(t)) = e^{itK_1}$ and define

$$\Delta_{\mathcal{W}_R}^{1/2} \doteq e^{\pi K_1} \, , \; J_{\mathcal{W}_R} \doteq U_1(\theta_R)$$

$$S_{\mathcal{W}_R} \doteq U_1(\theta_R) e^{\pi K_1} = J_{\mathcal{W}_R} \Delta_{\mathcal{W}_R}^{1/2}$$

$$\mathcal{K}_{\mathcal{W}_R} \doteq \{ f \in D(S_{\mathcal{W}_R}) \mid S_{\mathcal{W}_R} f = f \}$$

(real subspace of $\mathcal{H}_1$) Similarly for all $\mathcal{W} \in \boldsymbol{\mathcal{W}}$.

Theorem 2 (Brunetti, Guido & Longo, 2002). *Let* $\mathcal{W}_1 \subset \mathcal{W}_2$ *be wedges. Then* $\mathcal{K}_{\mathcal{W}_1} \subset \mathcal{K}_{\mathcal{W}_2}$ *if and only if* $U(\mathcal{P}_+)$ *satisfies the spectrum condition.*

$$e^h \doteq \oplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} h^{\otimes n} \, , \, h \in \mathcal{H}_1$$

$$V(f)e^0 \doteq e^{-\frac{1}{4}\|f\|^2} e^{\frac{i}{\sqrt{2}} f} \, , \, f \in \mathcal{H}_1$$

$$V(f)V(g) \doteq e^{-\frac{i}{2}\mathrm{Im}\langle f,g\rangle} V(f+g) \, , \, f, g \in \mathcal{H}_1$$

(unitary operators on $\mathcal{H}$)

Define $\mathcal{A}(\mathcal{W}_R) \doteq \{V(f) \mid f \in \mathcal{K}_{\mathcal{W}_R}\}''$.

Theorem 3 (Brunetti, Guido & Longo, 2002)**.** *If* $U(\mathcal{P}_+)$ *satisfies the spectrum condition, then* $(U(\mathcal{P}_+^\uparrow), \mathcal{A}(\mathcal{W}_R))$ *satisfies the consistency conditions and determines a local, Poincaré covariant net* $\{\mathcal{A}(\mathcal{O})\}$*. The Fock vacuum vector* Ω *is cyclic for* $\mathcal{A}(\mathcal{W})$*, for any wedge* $\mathcal{W}$*, and for* $\mathcal{A}(\mathcal{C})$*, for any spacelike cone* $\mathcal{C}$*. In the case that* $U_1(\mathcal{P}_+)$ *is irreducible with half integer spin,* Ω *is also cyclic for* $\mathcal{A}(\mathcal{O})$*, for any double cone* $\mathcal{O}$*.*

Further results were obtained by Fassarella & Schroer (2002) and Mund, Schroer & Yngvason (2004) in the special case of massless, “infinite spin” representations U_1. In particular, Ω is not cyclic for double cone algebras. So there is no local quantum field associated with the net in this case. (Yngvason, 1970)

Important Observation If $U_1(\mathcal{P}_+)$ is irreducible with half integer spin, the above construction yields the net associated with the free quantum field with the given mass and spin. If $U_1(\mathcal{P}_+)$ is the massless, “infinite spin” representation, then the net $\{\mathcal{A}(\mathcal{O})\}$ obtained above cannot be constructed from a local quantum field in the usual manner. (In fact, Mund, Schroer & Yngvason showed that the cone–localized field which “generates” the net cannot be expressed as an integral of a Wightman field along a suitable path.)

Factorizing S–Matrix Models — Using Nonlocal Fields to Define Local Nets

Inverse scattering theory: Given S, does there exist a decent quantum field model whose S–matrix is S?

$\hookrightarrow$ form factor program for integrable theories in $d = 2$

Schroer (1997, 1999): certain models of this type are amenable to methods developed in AQFT, including modular techniques

Lechner (2003–2008): implemented these ideas for a large class of two body scattering functions $S_2(\theta)$

$\mathcal{H}_1 = L^2(\mathbb{R})$, $\mathcal{H}$ is the S_2–symmetrized Fock space:

$$\Psi_n(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n) = S_2(\theta_1 - \theta_{i+1})\Psi_n(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n)$$

On $\mathcal{H}$ act creation, annihilation operators $Z^\dagger$, Z satisfying the Fadeev–Zamolodchikov relations:

$$Z^\dagger(\theta)Z^\dagger(\theta') = S_2(\theta - \theta')Z^\dagger(\theta')Z^\dagger(\theta)$$

(similarly for Z)

$$Z(\theta)Z^\dagger(\theta') = S_2(\theta' - \theta)Z^\dagger(\theta')Z(\theta) + \delta(\theta - \theta') \cdot \mathbf{1}$$

Quantum field:

$$\phi(f) \doteq Z^\dagger(f_+) + Z(f_-)\,,\; f \in \mathcal{S}(\mathbb{R}^2)\,,$$

where

$$f_\pm(\theta) = \frac{1}{2\pi}\int f(x)e^{\pm ip(\theta)x}\,dx$$

and $p(\theta) = m(\cosh\theta, \sinh\theta)$.

On $\mathcal{H}$ acts a canonical representation $U(\mathcal{P}_+^\uparrow)$ satisfying the spectrum condition.

Theorem 4 (Lechner, 2003). ϕ *transforms covariantly under* $U(\mathcal{P}_+^\uparrow)$,

$$(\Box + m^2)\phi(f) = 0\,,\ \forall f$$

$[\phi(x), \phi(y)] = 0$ *for all spacelike separated* x, y *if and only if* $S_2 \equiv 1$.

Defining $\mathcal{A}(\mathcal{W}) \doteq \{e^{i\phi(f)} \mid \text{supp}(f) \subset \mathcal{W}\}''$, one has $\mathcal{A}(\mathcal{W}) \not\subset \mathcal{A}(\mathcal{W}')'$ unless $S_2 \equiv 1$.

Recall that for $d = 2$ double cones $\mathcal{O} = \mathcal{W}_1 \cap \mathcal{W}_2'$ with $\mathcal{W}_2 \subset \mathcal{W}_1$. If $\{\mathcal{A}(\mathcal{W})\}$ were local, we would define

$$\mathcal{A}(\mathcal{O}) \doteq \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2') \, .$$

Here, we must consider

$$\mathcal{A}(\mathcal{O}) \doteq \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2)' \, ,$$

for then $\mathcal{O}_1 \subset \mathcal{O}_2'$ entails $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$. Indeed, one has in the indicated geometric situation $\mathcal{A}(\mathcal{W}_{12}) \supset \mathcal{A}(\mathcal{W}_{21})$, which implies $\mathcal{A}(\mathcal{W}_{12})' \subset \mathcal{A}(\mathcal{W}_{21})'$, so that

$$\begin{aligned} \mathcal{A}(\mathcal{O}_1) &= \mathcal{A}(W_{11}) \cap \mathcal{A}(\mathcal{W}_{12})' \subset \mathcal{A}(\mathcal{W}_{12})' \\ &\subset \mathcal{A}(\mathcal{W}_{21})' \subset \mathcal{A}(\mathcal{W}_{21})' \vee \mathcal{A}(\mathcal{W}_{22}) \\ &= \mathcal{A}(\mathcal{O}_2)' \, . \end{aligned}$$

Theorem 5 (Lechner, 2008)**.** *For a large class of S_2, Ω is cyclic and separating for all double cone algebras, the Haag–Ruelle scattering theory can be applied yielding an asymptotically complete scattering with S–matrix as prescribed.*

Though this approach is useful for $d = 2$, for $d = 4$ it results in theories with trivial scattering (Borchers, Buchholz & Schroer, 2001).

Important Lesson The construction is implemented using easily constructed nonlocal fields to obtain nonlocal wedge algebras $\mathcal{A}(\mathcal{W})$. Arguments (Smirnov (1992), Schroer (1999)) and examples (McCoy, Tracy & Wu (1977)) indicate that the associated local fields must be infinite power series in the (simple) nonlocal fields.

Buchholz & S. studied a special case ($S_2 \equiv -1$) of these models for $d \geq 2$. Indeed, acting on antisymmetric Fock space is a scalar Fermi field ϕ satisfying (Jost, 1964)

$$\phi\big((\Box + m^2)f\big) = 0\,,$$

and

$$\big\{\phi(f), \phi(g)\big\} \doteq \phi(f)\phi(g) + \phi(g)\phi(f) = \big(\langle\overline{g}|f\rangle + \langle\overline{f}|g\rangle\big) \cdot \mathbf{1}\,.$$

Both $[\phi(x), \phi(y)]$ and $\{\phi(x), \phi(y)\}$ are nonvanishing, even for spacelike separated x and y.

These fields are used to obtain a (maximally) nonlocal wedge net $\{\mathcal{A}(\mathcal{W})\}_{\mathcal{W} \in \boldsymbol{\mathcal{W}}}$.

For $d = 4$, such an $\mathcal{O} = \mathcal{W}_1 \cap \mathcal{W}_2{}'$ is unbounded (spacelike cylinder). Defining

$$\mathcal{A}_0(\mathcal{O}) \doteq \mathcal{A}(\mathcal{W}_1) \cap \mathcal{A}(\mathcal{W}_2)' \, ,$$

one has

Theorem 6 (Buchholz & S., 2007)**.** *For $d = 4$, the net $\{\mathcal{A}_0(\mathcal{O})\}$ of spacelike cylinder algebras determined by any fixed coherent family $\boldsymbol{\mathcal{W}}_0$ of wedges is local, nontrivial and is covariant under $U(\mathcal{P}_0)$, where $\mathcal{P}_0$ is the largest subgroup of $\mathcal{P}_+$ leaving the set $\boldsymbol{\mathcal{W}}_0$ fixed. Two body scattering can be defined, but is trivial. For $d = 2$, Haag–Ruelle scattering theory is applicable, and one has*

$$S = (-\mathbf{1})^{N(N-1)/2} \, .$$

Deformations

Grosse & Lechner (2007, 2008), Buchholz & S. (2008, 2009) ($d \geq 2$)

In order to construct concrete models on noncommutative Minkowski space, Grosse and Lechner (2007) performed a certain deformation upon the free quantum field on Minkowski space. They remarked that the resultant net could be interpreted either on noncommutative Minkowski space or on Minkowski space. Buchholz and S. realized that the deformation of Grosse and Lechner was a special case of a deformation which could be applied to any model, not just the free field.

Begin with an initial net $\{\mathcal{A}(\mathcal{O})\}$ and representation $U(\mathcal{P}_+^\uparrow)$ satisfying the spectrum condition. Consider the set $\mathfrak{F}$ of all operators F which are smooth under the adjoint action $\alpha_x(F) \doteq U(x)FU(x)^{-1}$, $x \in \mathbb{R}^4$. With

$$U(x) = e^{iPx} = \int e^{ipx}\, dE(p)\,, \quad x \in \mathbb{R}^4\,,$$

and any skew–symmetric 4×4–matrix Q, define for any $F \in \mathfrak{F}$

$$_QF \doteq \int \alpha_{Qp}(F)\, dE(p)\,, \quad F_Q \doteq \int dE(p)\, \alpha_{Qp}(F)\,. \tag{1}$$

The operators $_QF$ and F_Q are typically unbounded, even if F is bounded.

Theorem 7 (Buchholz & S., 2008)**.** *For all* $F \in \mathfrak{F}$ *one has:*

(a) ${}_QF = F_Q$

(b) ${F^*}_Q \subset {F_Q}^*$

(c) $(F_{Q_1})_{Q_2} = F_{Q_1+Q_2}$ *for all skew–symmetric* Q_1, Q_2

(d) $\alpha_\lambda(F_Q) = \big(\alpha_\lambda(F)\big)_{\Lambda Q \Lambda^{-1}}$ *for all* $\lambda \in \mathcal{P}_+^\uparrow$

(e) ${}_QF\Omega = F_Q\Omega = F\Omega$

$$F_Q = \int dE(p)\, \alpha_{Qp}(F) = \int dE(p)\, U(Qp) F U(Qp)^{-1} \int dE(q)$$
$$= \int\!\!\int dE(p)\, U(Qp) F\, U(Qp)^{-1}\, dE(q)$$
$$= \int\!\!\int dE(p)\, F e^{ipQq}\, dE(q)$$
$$= \int\!\!\int dE(p)\, e^{ipQq} F\, dE(q)$$
$$= \int\!\!\int dE(p)\, U(Qq) F U(Qq)^{-1}\, dE(q)$$
$$= \int \alpha_{Qq}(F)\, dE(q) = {}_Q F\,.$$

Uses $\int dE(q) = 1$; the skew symmetry of Q, implying $dE(p)\, e^{+iPQp} = dE(p)$ and $e^{-iPQp}\, dE(q) = e^{-iqQp} dE(q) = e^{ipQq} dE(q)$; e^{ipQq} is a c–number; $dE(p)\, e^{ipQq} = dE(p)\, e^{iPQq}$ and $dE(q) = e^{-iPQq}\, dE(q)$.

Now consider Q which, with respect to the chosen proper coordinates (those for which $\mathcal{W}_0 = \mathcal{W}_R$), has the form

$$Q \doteq \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \\ 0 & 0 & -\rho & 0 \end{pmatrix}$$

for some fixed $\kappa > 0$, $\rho \in \mathbb{R}$. Note that this matrix is skew symmetric with respect to the Lorentz inner product.

For such Q's Grosse and Lechner (2007) observed

(i) Let $\lambda = (\Lambda, x) \in \mathcal{P}_+^\uparrow$ be such that $\lambda \mathcal{W}_0 \subset \mathcal{W}_0$. Then $\Lambda Q \Lambda^{-1} = Q$.

(ii) Let $\lambda' = (\Lambda', x') \in \mathcal{P}_+^\uparrow$ be such that $\lambda' \mathcal{W}_0 \subset \mathcal{W}_0{}'$. Then $\Lambda' Q \Lambda'^{-1} = -Q$.

(iii) $QV_+ = \mathcal{W}_0$.

Definition 0.1. *Let* $\mathcal{W} \in \boldsymbol{\mathcal{W}}$ *and let* $\lambda = (\Lambda, x) \in \mathcal{P}_+^\uparrow$ *be such that* $\mathcal{W} = \lambda \mathcal{W}_0$*. Let* $\mathfrak{A}_Q(\mathcal{W})$ *be the polynomial algebra generated by all warped operators* $A_{\Lambda Q \Lambda^{-1}}$ *with* $A \in \mathfrak{A}(\mathcal{W}) \doteq \mathcal{A}(\mathcal{W}) \cap \mathfrak{F}$*.*

Theorem 8 (Buchholz & S., 2008, 2009)**.** *Let* $\mathcal{A}(\mathcal{W})$*,* $\mathcal{W} \in \boldsymbol{\mathcal{W}}$*, be a family of wedge algebras having the Reeh–Schlieder property and satisfying the conditions of isotony, covariance, and locality. Then the family of deformed algebras* $\mathfrak{A}_Q(\mathcal{W}) \subset \mathfrak{F}$*,* $\mathcal{W} \in \boldsymbol{\mathcal{W}}$*, also has these properties.*

The $\mathfrak{A}_Q(\mathcal{W})$ are *–algebras of unbounded operators.

However, it turns out that if $A = A^* \in \mathcal{A}(\mathcal{W})$ and $f = \overline{f}$ is a test function with supp$(f) \subset \mathcal{W}$ — so that

$$F \doteq A(f) \doteq \int \alpha_x(A)\, f(x)\, dx \in \mathfrak{F} \cap \mathcal{A}(\mathcal{W})$$

— then F_Q is (essentially) selfadjoint. Moreover, if F_1 and F_2 are such operators localized in $\mathcal{W}_R$, resp. $\mathcal{W}_R{}'$, then the spectral projections of $(F_1)_Q$ and $(F_2)_{-Q}$ commute.

Theorem 9 (Buchholz & S., 2009)**.** *Under the same assumptions as above, if $\mathcal{A}_Q(\mathcal{W}_R) \doteq \{A_Q \mid A \in \mathfrak{A}(\mathcal{W}_R)\}''$, then $\mathcal{A}_Q(\mathcal{W}_R)$ satisfies the consistency conditions, resulting in a local and covariant net $\{\mathcal{A}_Q(\mathcal{O})\}$.*

Scattering Theory

Moreover, if the original theory describes a scalar massive particle, then two body scattering is well defined in the deformed theory. Pick $A \in \mathfrak{A}(\mathcal{W}_0)$, $A' \in \mathfrak{A}(\mathcal{W}_0{}')$ interpolating between Ω and single particle states of mass m. Consider $A_Q \in \mathfrak{A}_Q(\mathcal{W}_0)$, $A'_{-Q} \in \mathfrak{A}(\mathcal{W}_0{}')$ and recall $A_Q\Omega = A\Omega$, $A'_{-Q}\Omega = A'\Omega$. Smearing these with test functions having suitable energy–momentum support (Haag–Ruelle), we have

$$|A(f)\Omega \otimes_Q A'(f')\Omega\rangle^{\text{in}} \doteq \lim_{t\to-\infty} A_Q(f_t)A'_{-Q}(f'_t)\Omega$$

$$|A(f)\Omega \otimes_Q A'(f')\Omega\rangle^{\text{out}} \doteq \lim_{t\to\infty} A_Q(f_t)A'_{-Q}(f'_t)\Omega\,,$$

if f, f' have suitable space–time supports. The limit vectors have all properties of a symmetric tensor product of the single particle states $A(f)\Omega$, $A'(f')\Omega$. In particular, they do not depend on the specific choice of operators A, A' and test functions f, f' within the appropriate limitations. These vectors form a basis in the respective asymptotic two–particle spaces.

This results in the following relation between the deformed scattering states and the original scattering states:

$$|p \otimes_Q q\rangle^{\text{in}} = e^{i|pQq|} \, |p \otimes q\rangle^{\text{in}}$$
$$|p \otimes_Q q\rangle^{\text{out}} = e^{-i|pQq|} \, |p \otimes q\rangle^{\text{out}} \, .$$

Note that the deformed scattering states depend upon the choice of $\mathcal{W}_0$ through the choice of Q and thus break the Lorentz symmetry.

The kernels of the elastic scattering matrices in the deformed and undeformed theory are related by

$$^{out}\langle p \otimes_Q q | p' \otimes_Q q' \rangle^{in} = e^{i|pQq| + i|p'Qq'|} \;\; ^{out}\langle p \otimes q | p' \otimes q' \rangle^{in} \, .$$

Thus they differ from each other, showing that the initial and deformed theories are not isomorphic.

Observables With Localizations Smaller Than Wedges?

- Algebras associated with double cones are trivial.
- At least for a one parameter family of admissible Q's, algebras associated with spacelike cylinders are nontrivial.

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- A nontechnical overview of constructive AQFT can be found at http://www.math.ufl.edu/ sjs/construction.html, where in the blank space is to be written a tilde.