

Nuclear Typing of Frames vs Spatial Selectors

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ABSTRACT. Nuclei which are defined over a class of frames are called nuclear typings. There is the dual notion of a spatial selector, and the relationship between nuclear typings and spatial selectors emanates from the duality between spatial frames and sober spaces. Especially interesting is the interplay between typings that are well-behaved with respect to certain frame quotients and selectors which similarly behave well in passage to closed sets.

Introduction

Motivated by applications to ℓ -groups, the articles [M04a, MZ04a] develop a concept of dimension of algebraic frames. A starting point was the idea of Krull dimension, i.e., the length of the longest chain of prime ideals in a commutative ring. Unfortunately, Krull dimension of the frame of convex ℓ -subgroups has limited relevance in the study of ℓ -groups. For reasonably important ℓ -groups, such as the ring $C(X)$ of real valued functions on a Tychonoff space, have very long and dense chains of prime convex ℓ -subgroups: nontrivial chains of prime ideals are η_1 sets! (See [GJ76].) Nonetheless, interesting notions of dimension are developed, by passing to a suitable frame quotient of the frame of convex ℓ -subgroups; this amounts to considering chains of “selected” primes, such as the prime z -ideals in $C(X)$.

One is led then to the notion of a “suitable quotient of an algebraic frame”; this raises interesting topological and frame theoretic issues, which this article aims to address. By the standard duality between sober spaces and spatial frames, a rule j which selects a quotient $j_A : A \rightarrow JA$ of each frame A roughly corresponds to a rule σ selecting a subspace of each space. The former is the nuclear typing of the title of this paper, the latter the spatial selector. Intuitively, if B is a suitable frame quotient of A , then the dimension of the former should not exceed the dimension of the latter. Thus, one is led to examine j -natural maps, i.e., frame maps $f : A \rightarrow B$ that induce maps $J(f) : JA \rightarrow JB$ such that

$$J(f) \cdot j_A = j_B \cdot f.$$

Roughly, the dual notion is the following: for a continuous function $f : Y \rightarrow X$ the question of when $f(\sigma(Y)) \subseteq \sigma(X)$. There is a stronger notion, which will play an

important role in applications to be considered elsewhere: a subspace $Y \subseteq X$ is σ -exact if $\sigma(Y) = Y \cap \sigma(X)$. There is also a notion of a j -exact quotient, designed to determine when the corresponding subspace diagram is exact.

Given our intended future applications, spatial frames and – in particular – algebraic frames, play a key role. Thus, we make repeated use of “inductivization”. Given any nucleus on an algebraic frame there is a largest inductive nucleus below it. Dually, in an algebraic space, that is, a space with a base of compact open sets which is closed under binary intersections, the patch closure of a subspace S is the smallest algebraic subspace containing S , such that the inclusion map is coherent. These key facts, developed independently in [MZ03], and earlier, in a more general setting in [Es98, §3] and [Es01], allow considerable flexibility in defining subspace selectors and nuclear typings.

In this article, the authors were motivated primarily by the subject of dimension in ℓ -groups. However, the notion of typings seems to have relevance to other interesting ℓ -group questions, which will be discussed elsewhere. This discussion involves ℓ -group extensions with special features; such an extension $G \subseteq H$ induces a frame embedding of the frame of convex ℓ -subgroups of G in the frame of convex ℓ -subgroups of H . Abstracting to frames, consider a nuclear typing j . The setting of ℓ -extension alluded to above can be captured at the level of frames by considering j -natural frame embeddings $h : A \rightarrow B$. More pointedly, it is of interest to determine when the induced $J(h)$ is injective, surjective, or an isomorphism.

§1 provides definitions and some background needed in the following; readers familiar with frame theory, or one of our recent articles on dimension may choose to skip it, referring back when necessary to understand later developments. §2 develops the notion of a nuclear typing on a category of frames: the key concepts are j -naturality and j -exactness. In §3 we discuss the dual notion of a spatial selector; this is also where the general “pairing” theorems between typings and selectors are presented (Proposition 3.7 and Theorem 3.8). In §4 the duality between typings and selectors is studied in algebraic frames and algebraic spaces, and it is shown (Proposition 4.8) that inductivization corresponds to the spatial selector which defines patch closure. Finally, there is a wealth of examples in §5, which ought to both illustrate the theory, and prompt the enterprising reader to investigate further.

1 Frame-theoretic Preliminaries.

This section contains a brief dictionary of frame-theoretic terms, followed by an equally brief account of the adjointness and duality between frames and topological space. We refer the reader to [J82] and [JT84] for additional general background on frames. Our basic reference for category theory is [HS79].

Definition & Remarks 1.1. Throughout this commentary, L is a complete lattice. The top and bottom are denoted 1 and 0 , respectively. For each $x \in L$, $\downarrow x$ (resp. $\uparrow x$) denotes the set of elements of L which are $\leq x$ (resp. $\geq x$).

1. $a \in L$ is *compact* if $a \leq \bigvee S$ implies that $a \leq \bigvee F$, for a suitable finite subset F of S . The lattice L is *compact* if 1 is compact.
2. L is a *frame* if the following distributive law holds:

$$a \wedge \left(\bigvee S \right) = \bigvee \left\{ a \wedge s : s \in S \right\},$$

for each $a \in L$ and each $S \subseteq L$.

In the categories of frames used in this paper, the morphisms are always maps which preserve finite meets and arbitrary joins (and therefore, also top and bottom); these are the *frame homomorphisms* or *frame maps*.

3. $p < 1$ in L is *prime* if $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. $\text{Spec}(L)$ shall denote the set of prime elements of L . Unless the contrary is expressed, $\text{Spec}(L)$ is considered as a topological space under the *hull-kernel topology*. And for any subset S of $\text{Spec}(L)$, the subsets

$$c_S(x) = \{ s \in S : x \not\leq s \} \quad (\text{with } x \in L),$$

are the open sets of S in the subspace topology; $z_S(x)$ designates the complement of $c_S(x)$ in S .

4. Let L be a frame. For each $a, b \in L$, let

$$a \rightarrow b = \bigvee \left\{ x \in L : a \wedge x \leq b \right\}.$$

When $b = 0$ we denote $a \rightarrow 0 = a^\perp$. $x \in L$ is a *polar* if it is of the form $x = y^\perp$, for some $y \in L$. It is well known that the set PL of all polars forms a complete boolean algebra, in which infima agree with those in L .

5. Let L be a frame. Recall that $a \preceq b$ if $b \vee a^\perp = 1$; $x \in L$ is *regular* if

$$x = \bigvee \left\{ a \in L : a \preceq x \right\}.$$

L is *regular* if each element of L is regular.

6. A frame L is *normal* if for each decomposition $1 = a \vee b$, there exist disjoint c and d in L , such that $a \vee d = c \vee b = 1$.

7. (See [JT84].) Let L be a frame and suppose that $j : L \longrightarrow L$ is a closure operator; jL designates $\{x \in L : j(x) = x\}$. j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$. It is well known that j is a nucleus if and only if $b \in jL$ implies that $a \rightarrow b \in jL$, for each $a \in L$. For convenience we shall call a subset with this feature *nuclear*. The members of jL will be referred to as *j -elements*.

The map $p(x) = x^{\perp\perp}$ defines a nucleus for which the fixed subset (in L) is PL .

8. The nucleus j is *dense* if $j(0) = 0$; if so we also say that L is *j -semisimple*. Note that j is dense if and only if $0 \in jL$.

Next, we wish to make reference to the adjunction between, on the one hand, the category \mathfrak{Frm} of all frames and all frame maps, and, on the other hand, the category \mathfrak{Top} of all topological spaces and all continuous functions. For details we refer the reader to [J82]. The background in category theory may be found in [HS79, Chapter VII].

Remarks 1.2. There is an adjoint pair of contravariant functors

$$\text{Spec} : \mathfrak{Frm} \longleftarrow \mathfrak{Top} : \mathfrak{D},$$

where $\mathfrak{D}(X)$ denotes the frame of open sets of the topological space X . The two units of the adjunctions are $\text{coz} : 1_{\mathfrak{Frm}} \longrightarrow \mathfrak{D} \cdot \text{Spec}$, defined by

$$\text{coz}_L(x) \equiv c_{\text{Spec}(L)}(x) = \{p \in \text{Spec}(L) : x \not\leq p\},$$

and $\varepsilon : 1_{\mathfrak{Top}} \longrightarrow \text{Spec} \cdot \mathfrak{D}$, defined by

$$\varepsilon_X(p) \equiv X \setminus \text{cl}_X\{p\},$$

with $\text{cl}_X Y$ denoting the closure of Y in X .

The principal features of this adjunction are:

1. coz amounts to an epireflection ([HS79, §37]) of \mathfrak{Frm} in $\mathfrak{S}\mathfrak{Frm}$, the full subcategory of \mathfrak{Frm} consisting of all *spatial* frames.
2. ε_X is one-to-one precisely when the space X is a T_0 *space* – that is, given two distinct points of X , there is an open set containing one point but not the other.
3. The adjunction induces a duality of categories: between $\mathfrak{S}\mathfrak{Frm}$ and $\mathfrak{S}\mathfrak{Top}$, the full subcategory of \mathfrak{Top} consisting of all *sober* spaces. Thus, the sober spaces are precisely the spaces of the form $\text{Spec}(L)$, for a suitable frame L , while a frame is spatial if and only if it is isomorphic to $\mathfrak{D}(X)$, for some space X . Viewed “internally”, a frame L is spatial if and only if each $x \in L$ is a meet of primes of L .

4. Note that a space X is sober if and only if it is a T_0 space and each irreducible closed set is the closure of a singleton set. Recall that the closed $K \subseteq X$ is *irreducible* if $K \subseteq A \cup B$ (with both A and B closed) implies that $K \subseteq A$ or $K \subseteq B$.

2 Nuclear Typings.

The ambient category throughout this section is \mathfrak{Frm} . In this section the formal study of nuclei on a subclass or subcategory of \mathfrak{Frm} begins. Whereas the consideration of the morphisms might, generally, appear to be of secondary importance in the discussion below, the reader should remember that the introduction of these nuclear typings is motivated by interest in frame quotients.

Definition & Remarks 2.1. (a) Fix a full subcategory \mathfrak{C} of \mathfrak{Frm} . A *nuclear typing* \mathfrak{C} is a function j which associates with each \mathfrak{C} -object A , a nucleus j_A on A . For brevity we denote the image of j_A as JA . Let us underscore: one may think of a nuclear typing as a pair (j, J) , such that, for each \mathfrak{C} -object A , j_A is a nucleus with fixed set JA . We will keep to this distinction, using lower case letters for the nucleus and upper case letters for the target frame, throughout for the theorems and propositions. There are some exceptions to this usage, such as in Examples 2.12, 5.4, and 5.5, for which the “lower-case” convention (exclusively) is well-established in the literature, or in situations such as Theorem 3.8(d), where the notation makes the distinction absurd.

Let $h : A \rightarrow B$ be a \mathfrak{C} -morphism; h is said to be *j -natural* when there is a frame homomorphism $J(h) : JA \rightarrow JB$, for which the following diagram commutes:

$$(2.1.1) \quad \begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow j_A & & \downarrow j_B \\ JA & \xrightarrow{J(h)} & JB \end{array}$$

that is, $J(h) \cdot j_A = j_B \cdot h$. Observe as well, that $J(h)$ is uniquely determined by h .

\mathfrak{C}_j denotes the subcategory of \mathfrak{C} -objects with all j -natural maps. The reader will easily verify that the composition of j -natural maps is j -natural, so that \mathfrak{C}_j is, indeed, a subcategory of \mathfrak{C} .

- (b) The map $J(h)$ in (a) is well defined if and only if $j_A(x) = j_A(y)$ implies that $j_B(h(x)) = j_B(h(y))$; this is obvious, but useful when studying examples.

Three examples are given here. They are representative of the nuclear typings we have in mind. We begin with the nuclear typing of polars p .

Example 2.2. In 1.1.7 we defined $p(x) = x^{\perp\perp}$, for $x \in L$. This obviously defines a nuclear typing on \mathfrak{Frm} , called the *polar typing*.

The frame homomorphism $h : A \rightarrow B$ is p -natural precisely when

$$(\Delta) \quad p_B(h(p_A(x))) = p_B(h(x)),$$

for each $x \in A$. It is easy to verify that (Δ) is equivalent to each of the following conditions:

- (i) $x_1^{\perp} = x_2^{\perp} \Rightarrow h(x_1)^{\perp} = h(x_2)^{\perp}$.
- (ii) Whenever $x \in A$ is dense, then $h(x)$ is dense in B . ($x \in A$ is *dense* if $x^{\perp} = 0$.)

A model for the next example in commutative algebra is the Jacobson radical.

Example 2.3. For each frame L , let $\text{Max}(L)$ denote the set of all maximal elements, which could be empty. In anticipation of notation to be introduced in later, we designate

$$\text{Max}^*(x) = \bigwedge \{ m \in \text{Max}(L) : x \leq m \}.$$

It is easy to verify that Max^* defines a nucleus, and therefore a nuclear typing. We shall refer to Max^* in the sequel as the *Max typing*.

One of the basic ingredients of a suitable “dimension theory” in algebraic frames, as the concept will be developed in [MZ04∞], is that for each frame L under consideration, and certain members x of L , the dimension of $\uparrow x$ not exceed that of L . At the level of frames, and in the context of a nuclear typing j , it is important, in this regard, to examine the j -naturality of the frame map $h_A^t : A \rightarrow \uparrow t$, defined by

$$h_A^t(x) = x \vee t, \quad t \in A.$$

In view of the discussion in 2.1, one might expect the usefulness of compatibility between j_A and $j_{\uparrow t}$.

Definition 2.4. Assume that j is a nuclear typing on \mathfrak{C} , and that for each \mathfrak{C} -object A and each $t \in A$, h_t^A is a \mathfrak{C} -morphism. We say that $t \in A$ is *j -exact* if for each $y \geq t$,

$$j_A(y) = j_{\uparrow t}(y).$$

Frame cognoscenti will easily see that $t \in A$ is j -exact if and only if the square (2.1.1) is a pushout in the category of frames.

As we shall see below, every polar is p -exact, but this is not necessarily true of every element of the frame, whereas every member of the frame is Max^* -exact.

The following result is reasonably straightforward.

Proposition 2.5. *Suppose that j is a nuclear typing on the subcategory \mathfrak{C} , and that h_A^t is a \mathfrak{C} -morphism. Then $t \in A$ is j -exact if and only if*

- (i) h_A^t is j -natural and
- (ii) $J(h_A^t) = h_{JA}^{j_A(t)}$.

Proof. Assume first that t is j -exact. Note that if $x, y \geq t$ and $j_A(x) = j_A(y)$, then we also have that $j_{\uparrow t}(x) = j_{\uparrow t}(y)$, whence h_A^t is j -natural. In fact, for each $a \in A$,

$$J(h_A^t)(j_A(a)) = j_{\uparrow t}(h^t(a)) = j_{\uparrow t}(a \vee t) = j_A(a \vee t) = j_A(a) \vee^{JA} j_A(t),$$

where \vee^{JA} denotes the join in JA . The upshot of the above calculation is that $J(h_A^t) = h_{JA}^{j_A(t)}$, as claimed. Conversely, if h_A^t is j -natural, and $J(h_A^t) = h_{JA}^{j_A(t)}$, then the j -exactness of t follows from an easy calculation, which we leave to the reader. ■

In Proposition 2.5, the j -naturality of h_A^t does not imply that t is j -exact. That is, condition (ii) in that proposition does not follow from condition (i). We refer the reader to Example 5.16.

Proposition 2.6. *Every polar is p -exact.*

Proof. We begin with the observations below, for any $a, t \in L$, leaving the proof to the reader. Note: b' denotes the polar of $b \geq t$ in $\uparrow t$.

1. $(a \vee t)' = (a \vee t) \rightarrow t$, and
2. $(a \vee t)'' = (a \vee t)' \rightarrow t$.
3. Furthermore, $p(a) \leq (a \vee t)''$.

We now show that each polar is p -exact. To that end, suppose that $t \in pL$ and $y \geq t$; we show that $y' = y^\perp \vee^{PL} t$, which is sufficient. But observe that

$$x \leq y' \iff x \wedge y \leq t \iff x \wedge y \wedge t^\perp = 0 \iff x \leq (y \wedge t^\perp)^\perp,$$

and $(y \wedge t^\perp)^\perp = y^\perp \vee^{PL} t$, by de Morgan's Law, and the fact that t is a polar. ■

Proposition 2.7. *Every $t \in L$ is Max^* -exact.*

Proof. Simply observe that, for each $t \in L$, $\text{Max}(L) \cap (\uparrow t) = \text{Max}(\uparrow t)$. ■

In Example 5.13 it is shown that not every element of a frame is p -exact. In Example 5.15 we exhibit a nuclear typing j and an element of jL which is not j -exact.

The section culminates with some observations designed to motivate the spatial selectors of the following section.

Remark 2.8. Suppose that $h : A \longrightarrow B$ is a morphism of complete join-semilattices. Define $h_* : B \longrightarrow A$ by the following equivalence (which defines it unambiguously):

$$x \leq h_*(y) \iff h(x) \leq y.$$

Thus, viewing A and B as categories, h_* is none other than the right adjoint of h ([HS79, 27.1]). As a consequence of the adjoint relationship, we have the the following properties:

1. $x \leq h_*(h(x))$, for each $x \in A$, and $h(h_*(y)) \leq y$, for each $y \in B$.
2. h_* preserves arbitrary infima.
3. Assuming that h is surjective as well, we have:
 - (a) $h \cdot h_* = 1_B$.
 - (b) $n(h) \equiv h_* \cdot h : A \longrightarrow B$ is a closure operator, and h_* induces an isomorphism of B onto $n(h)A$, the inverse of which is $h|_{n(h)A}$.
4. Let us now return to a frame map between \mathfrak{Frm} -objects, $h : A \longrightarrow B$. Denote $h_*|_{\text{Spec}(B)} \equiv \text{Spec}(h)$. Then $n(h)$ is a nucleus and $\text{Spec}(h)$ is a continuous map $\text{Spec}(B) \longrightarrow \text{Spec}(A)$. If h is a j -natural surjection, then $\text{Spec}(h)$ is an embedding of subspaces. Consider now the square, induced by diagram (2.1.1):

$$(2.8.1) \quad \begin{array}{ccc} \text{Spec}(A) & \xleftarrow{\text{Spec}(h)} & \text{Spec}(B) \\ \uparrow \text{Spec}(j_A) & & \uparrow \text{Spec}(j_B) \\ \text{Spec}(JA) & \xleftarrow{\text{Spec}(J(h))} & \text{Spec}(JB) \end{array}$$

The diagram represents embeddings of subspaces of $\text{Spec}(A)$. The vertical embeddings are the inclusions of $\text{Spec}(JA)$ in $\text{Spec}(A)$ and $\text{Spec}(JB)$ in $\text{Spec}(B)$, respectively. That this diagram commutes should be obvious. What is sought in Proposition 2.9 below is a reasonable condition under which (2.8.1) is an *inverse-image diagram* (or, equivalently, a *pullback*); see [HS79, §21].

Proposition 2.9. *Suppose that j is a nuclear typing defined on the subcategory \mathfrak{C} of \mathfrak{Frm} , and let $h : A \rightarrow B$ be a surjective \mathfrak{C}_j -morphism. Regarding the commutative square (2.8.1):*

- (a) $h_*|_{JB} = j(h)_*$, and so $\text{Spec}(j(h))$ agrees with $\text{Spec}(h)$ on $\text{Spec}(JB)$.
- (b) $h_*^{-1}(JA) = JB \iff JA = h_*(JB)$, and if this is the case, then

$$\text{Spec}(h)^{-1}(\text{Spec}(JA)) = \text{Spec}(JB).$$

Proof. (a) For $x \in A$ and $y \in JB$, we calculate:

$$\begin{aligned} x \leq J(h)_*(y) &\iff j_A(x) \leq j(h)_*(y), && \text{as } j(h)_*(y) \in JA; \\ &\iff J(h)(j_A(x)) \leq y; \\ &\iff j_B(h(x)) \leq y, && \text{since } j(h)j_A = j_B h; \\ &\iff h(x) \leq y, && \text{since } j_B(y) = y; \\ &\iff x \leq h_*(y). \end{aligned}$$

It follows that $J(h)_*(y) = h_*(y)$, proving (a).

(b) follows easily from (a) and is left to the reader. ■

Definition & Remarks 2.10. Suppose that j is a nuclear typing defined on the subcategory \mathfrak{C} of \mathfrak{Frm} , and let $h : A \rightarrow B$ be a surjective \mathfrak{C}_j -morphism. If $h_*^{-1}(JA) = JB$, as in Proposition 2.9(b), we will say that h is *j -exact*.

As the reader no doubt expects, for the frame quotients h_A^t , the two notions of exactness coincide.

Proposition 2.11. *Suppose that j is a nuclear typing defined on the subcategory \mathfrak{C} of \mathfrak{Frm} . Assume that A is a \mathfrak{C} -object such that h_A^t is a \mathfrak{C}_j -morphism (with $t \in A$). Then h_A^t is j -exact precisely when t is j -exact.*

Proof. If t is j -exact, then $j_A(x) = j_{\uparrow t}(x)$ for each $x \geq t$. Thus, if $x \in j(\uparrow t)$, it follows that $j_A(x) = x$ as well, and $(h_A^t)_*$ satisfies condition (b) of Proposition 2.9; that is, h_A^t is j -exact.

Conversely, if h_A^t is j -exact, then since it is j -natural, it suffices to show that (ii) in Proposition 2.5 holds. As to that, note that $J(h_A^t)_*(x) = (h_A^t)_*(x) = x$, for each $x \in J(\uparrow t)$, according to Proposition 2.9(a). By the definition of adjoints (2.8), we have that $j(h_A^t)(a) \leq x$, for $x \in J(\uparrow t)$ and $a \in JA$, if and only if $a \leq x$ (with $x \geq j_A(t)$). That is,

$$j(h_A^t)(a) \leq x \iff a \vee^{jA} j_A(t) \leq x,$$

as desired. ■

Finally, in this section, another example, which ought to caution the reader about j -naturality. For details see [Ba97, Ba02].

Example 2.12. Assume that L is a compact frame. The *saturation nucleus* $s = s_L : L \longrightarrow sL$ is defined by

$$s(x) = \vee \{ a \in L : a \vee y = 1 \Rightarrow x \vee y = 1 \}.$$

Note that, in view of the compactness of 1, $s(x)$ is the largest element for which $s(x) \vee y = 1$ implies that $x \vee y = 1$. With sufficiently strong set-theoretic axioms – such as Zorn’s Lemma – the nucleus s coincides with Max^* (Example 3.5).

Now, regarded on the full subcategory $\mathfrak{KN}\mathfrak{Frm}$ of \mathfrak{Frm} consisting of all compact normal frames, the saturation nucleus is functorial ([Ba02, Proposition 2.3]), but not every frame map between compact normal frames is s -natural ([Ba02, 2.5]).

3 Spatial Selectors

This section looks at the topological counterpart of a nuclear typing. The correspondence between the spatial selectors to be defined below and the nuclear typings, as it is detailed in Proposition 3.7 and Theorem 3.8, should seem natural to the knowledgeable reader. The notion that nuclei are dual to subspaces is discussed explicitly in [J82] and [Es01].

$\text{Nuc}(L)$ designates the poset of all nuclei on a frame L , partially ordered, pointwise, that is, by $j_1 \leq j_2$ provided $j_1(x) \leq j_2(x)$, for each $x \in L$. The power set of a set X is denoted $\mathfrak{P}(X)$.

Remarks 3.1. First, define $*$: $\text{Nuc}(L) \longrightarrow \mathfrak{P}(\text{Spec}(L))$ by $j^* = jL \cap \text{Spec}(L)$. In the reverse direction we have the map $*$: $\mathfrak{P}(\text{Spec}(L)) \longrightarrow \text{Nuc}(L)$ defined by $S^*(x) = \wedge \{ p \in S : x \leq p \}$. Note that S^* is indeed a nucleus. It is routine to verify the following properties:

1. (a) $*$ is order reversing, in both directions.
- (b) Since j is a nucleus, we have that $\text{Spec}(jL) = jL \cap \text{Spec}(L) = j^*$.
- (c) $j \leq j^{**}$; (because $j(x) \leq \wedge \{ p \in j^* : x \leq p \} = j^{**}(x)$.)
- (d) $S \subseteq S^{**}$; (for, if $p \in S \subseteq \text{Spec}(L)$ then $S^*(p) = p$.)

In sum, the $*$ maps give rise to a Galois correspondence between nuclei and subspaces. Then it follows that $S^* = S^{***}$ and $j^* = j^{***}$.

2. Moreover we have the following observations regarding the maps $**$; (the reader is referred to [J02, C] for elaboration):
 - (a) Suppose that L is a frame and $j \in \text{Nuc}(L)$. Then $j = j^{**}$ if and only if jL is spatial. Moreover, the induced frame homomorphism $j^{**} : jL \longrightarrow j^{**}L$ is a model of the spatial reflection of jL .

- (b) Let $S \subseteq \text{Spec}(L)$; then, $S^{**} = S$ if and only if S is sober. Moreover, $S \mapsto S^{**}$ is a model of the reflection in sober spaces.
3. As long as $\text{Spec}(L)$ is a root system, the closure $S \mapsto S^{**}$ has an order-theoretical interpretation. S^{**} is the least subset of $\text{Spec}(L)$ containing S which is closed under infima of chains. This is easy to show directly, without appealing to the topological considerations above. We leave the details to the reader.

Otherwise, there is the following example. Let X be the space of all prime ideals of the ring of \mathbb{Z} integers under the hull-kernel topology. As is well known, X consists of the antichain M consisting of the ideals generated by prime numbers, together with the trivial ideal $\{0\}$, lying below every element of M . M is homeomorphic to \mathbb{N} with the finite complement topology, which is not sober, although it is T_1 , but not Hausdorff. On the other hand, $M^{**} = X \neq M$, whereas M is (trivially) closed under infima of chains.

With Proposition 2.9 in mind, we now introduce spatial selectors. The ambient category for this discussion is \mathfrak{STop} , the category of sober topological spaces (and continuous maps).

Definition & Remarks 3.2. (a) Let \mathfrak{A} be a subcategory of \mathfrak{STop} . A *spatial selector* is a rule which assigns to each space X in \mathfrak{A} a subspace σX . When the values of the spatial selector are endowed with some special topological property \mathcal{T} , we shall call σ a *\mathcal{T} selector*. The prototypical spatial selector is associated with a nuclear typing j (defined on a subcategory \mathfrak{C} of \mathfrak{Frm}), by stipulating that $\Sigma(j)(\text{Spec}(L)) = \text{Spec}(jL)$. The domain of $\Sigma(j)$ is the subcategory of \mathfrak{STop} consisting of the spectra of all objects in \mathfrak{C} ; we denote it $\Sigma\mathfrak{C}$.

We shall refer to $\Sigma(j)$ as the selector *induced by j* . Notice that the empty set may be a value of $\Sigma(j)$; see the discussion in Example 5.10. Please observe as well that it is not assumed that the values of a spatial selector on \mathfrak{A} -objects are again \mathfrak{A} -objects. Indeed, this fails to be so in most natural examples.

(b) Conversely, suppose that σ is a spatial selector on a subcategory \mathfrak{A} of \mathfrak{STop} . Then, for any \mathfrak{A} -object X , let $\mathcal{J}(\sigma)_{\mathfrak{D}(X)}$ be the *-dual $(\sigma X)^*$. Recall now – see 1.2 – that in the canonical homeomorphism $\varepsilon : X \longrightarrow \text{Spec}(\mathfrak{D}(X))$ the points $p \in X$ are “recovered” as the prime open sets $X \setminus \text{cl}_X\{p\}$. We then have the following explicit description of $\mathcal{J}(\sigma)_{\mathfrak{D}(X)}$; for each $U \in \mathfrak{D}(X)$,

$$\begin{aligned} \mathcal{J}(\sigma)_{\mathfrak{D}(X)}(U) &= \wedge \{ \varepsilon_X(p) : U \subseteq \varepsilon_X(p), p \in \sigma(X) \} \\ &= \text{int}_X [\bigcap \{ \varepsilon_X(p) : p \notin U, p \in \sigma(X) \}] \end{aligned}$$

The nuclear typing $\mathcal{J}(\sigma)$ is, likewise, said to be *induced by σ* . By symmetry with the notation in (a), let $\mathcal{J}\mathfrak{A}$ stand for the domain of the nuclear typing induced by σ .

(c) Let σ be a spatial selector on \mathfrak{A} , and S be a subspace of the space X , and that the inclusion $S \subseteq X$ is an \mathfrak{A} -morphism. We shall say that S is σ -exact if $\sigma S = \sigma X \cap S$. In this terminology, Proposition 2.9 states that if $A \longrightarrow B$ is a j -exact frame quotient, then $\text{Spec}(B)$ is a $\Sigma(j)$ -exact subspace of $\text{Spec}(A)$.

If every closed (resp. open, resp. clopen) embedding (belonging to \mathfrak{A}) is σ -exact, we say that σ is *closed-* (resp. *open-*, resp. *clopen-*) *hereditary*. Note that if S is a σ -exact subspace, and S is closed (resp. open, resp. clopen) then σS is closed (resp. open, resp. clopen) in σX .

More generally, if \mathcal{K} is a class of subspaces of X – with the feature that for each $S \in \mathcal{K}$ the embedding $S \subseteq X$ is an \mathfrak{A} -morphism, and each member of \mathcal{K} is σ -exact, we shall say that σ is *\mathcal{K} -hereditary relative to X* . We will drop the qualification “relative to X ” whenever the context is clear.

Here is a summary of the discussion in 3.2(b) above.

Proposition 3.3. *Let \mathfrak{A} be a category of sober spaces, σ be a spatial selector, and $j = \mathcal{J}(\sigma)$. For each \mathfrak{A} -object X and each $U \in \mathfrak{D}(X)$,*

$$j(U) = \text{int}_X[(X \setminus \sigma(X)) \cup U].$$

Alternatively, jU is the largest open set V such that $V \cap \sigma(X) \subseteq U \cap \sigma(X)$.

Proof. From the display in 3.2(b) we have, for each open set U :

$$j(U) = \text{int}_X [\cap \{ \varepsilon_X(p) : p \notin U, p \in \sigma(X) \}]$$

It is then evident that $x \in j(U)$ precisely when it has a neighborhood V contained in $(X \setminus \sigma(X)) \cup U$, that is, $j(U) = \text{int}_X[(X \setminus \sigma(X)) \cup U]$. The second claim easily follows from this identity. \blacksquare

Although the following corollary may be obtained independently, it seems worthwhile to record the observation.

Corollary 3.4. *Let \mathfrak{A} be a category of sober spaces, σ be a spatial selector, and $j = \mathcal{J}(\sigma)$. Let X be an \mathfrak{A} -object. If $t : \mathfrak{D}(X) \longrightarrow \mathfrak{D}(\sigma(X))$ denotes the trace map $t(U) = U \cap \sigma(X)$, then $t_* \cdot t = j$ (2.8.3).*

Before examining the properties of the maps $j \mapsto \Sigma(j)$ and $\sigma \mapsto \mathcal{J}(\sigma)$, and exploring the behavior of exactness vs. heredity under these correspondences, a few (canonical) examples are in order.

Example 3.5. On $\mathfrak{S}\mathfrak{T}\mathfrak{o}\mathfrak{p}$, consider $\text{Spec}(L) \mapsto \text{Max}(L)$ as a spatial selector. Note that $\text{Max}(L)$ could be empty, and that, while $\text{Max}(L)$ is a T_1 -space, it may fail to be sober – such as when L is the frame of semiprime ideals of the ring of integers.

As suggested by the sketch of the proof of Proposition 2.7, Max is a closed-hereditary selector. When Max is restricted to normal spaces, then its values are Hausdorff (5.2).

Example 3.6. For any frame L , $\text{Min}(L)$ stands for the set of all minimal primes. By Zorn's Lemma, each prime of L exceeds a minimal prime. We shall consider $\text{Spec}(L) \mapsto \text{Min}(L)$ as a spatial selector. As we shall see, Min is an open-hereditary Hausdorff selector, when suitably restricted.

Next, we examine the coupling of the maps $\sigma \mapsto \mathcal{J}(\sigma)$ and $j \mapsto \Sigma(j)$. The proof of the following result is straightforward; a sketch of the arguments suffices.

Proposition 3.7.

- (a) Let \mathfrak{A} be a subcategory of sober spaces, on which a spatial selector σ is defined. Then $\mathcal{J}(\Sigma(\mathcal{J}(\sigma))) = \mathcal{J}(\sigma)$. Let j be a nuclear typing on the subcategory \mathfrak{C} of \mathfrak{Frm} . Then $\Sigma(\mathcal{J}(\Sigma(j))) = \Sigma(j)$.
- (b) Let \mathfrak{A} be a subcategory of sober spaces, on which a spatial selector σ is defined. Suppose that X is an \mathfrak{A} -object. Then $\Sigma(\mathcal{J}(\sigma))(X) = \sigma(X)^{**}$. Consequently, $\sigma(X) \subseteq \Sigma(\mathcal{J}(\sigma))(X) \subseteq X$. Thus, $\Sigma \cdot \mathcal{J} = 1$ on the class of sober selectors.
- (c) Let j be a nuclear typing on the subcategory \mathfrak{C} of \mathfrak{Frm} . Then $j \leq \mathcal{J}(\Sigma(j))$, and equality holds if and only if the values of j are spatial frames; in this case, $\mathcal{J} \cdot \Sigma = 1$.

Proof. (a) This follows from the comments in 3.1.1.

(b) For each \mathfrak{A} -object X we have, by Proposition 3.3 and its corollary, together with 3.1.2b, that

$$\Sigma(\mathcal{J}(\sigma))(X) = \text{Spec}(\mathcal{J}(\sigma)\mathfrak{D}(X)) = \text{Spec}(\mathfrak{D}(\sigma(X))) = \sigma(X)^{**},$$

whence (b) follows.

(c) It should be clear that $j \leq \mathcal{J}(\Sigma(j))$; the rest follows from 3.1.2a. ■

We now examine exactness of elements of a frame vs exactness of closed sets. The result is the next theorem. For any nuclear typing j , let us agree to call each closed set of the form $(\uparrow t) \cap \text{Spec}(L)$, with $t \in jL$, a *j -closed set*.

Theorem 3.8.

- (a) Suppose that j is a nuclear typing on a subcategory \mathfrak{C} of \mathfrak{Frm} . If $t \in L$ is j -exact, then each j -closed set is $\Sigma(j)$ -exact. In particular, if each $t \in jL$ (resp. each $t \in L$) is j -exact, then $\Sigma(j)$ is j -closed-hereditary (resp. closed-hereditary).
- (b) Let \mathfrak{A} be a subcategory of sober spaces, and suppose that σ is a spatial selector on \mathfrak{A} . Assume that L is a $\mathcal{J}\mathfrak{A}$ -object and $t \in L$. If $(\uparrow t) \cap \text{Spec}(L)$ is σ -exact then t is $\mathcal{J}(\sigma)$ -exact. Thus, if σ is $\mathcal{J}(\sigma)$ -closed-hereditary (resp. closed-hereditary), then each $t \in \mathcal{J}(\sigma)L$ (resp. each $t \in L$) is $\mathcal{J}(\sigma)$ -exact.

- (c) Suppose that j is a nuclear typing on a subcategory \mathfrak{C} of \mathfrak{Ftm} , and that the values of j are spatial frames. Then each $t \in jL$ (resp. each $t \in L$) is j -exact if and only if $\Sigma(j)$ is j -closed-hereditary (resp. closed-hereditary).
- (d) Let \mathfrak{A} be a subcategory of sober spaces, and suppose that σ is a sober selector on \mathfrak{A} . Then σ is $\mathcal{J}(\sigma)$ -closed-hereditary (resp. closed-hereditary), if and only if each $t \in \mathcal{J}(\sigma)L$ (resp. each $t \in L$) is $\mathcal{J}(\sigma)$ -exact.

Proof. The claims in (c) and (d) evidently follow from (a) and (b), together with Proposition 3.7. We proceed to prove (a) and (b). The second part of the assertions in (a) and (b) obviously follows from the first, so we need only prove the first.

(a) We apply Proposition 2.9 to the frame quotient h_L^t . It guarantees that if h_L^t is j -exact then $\text{Spec}(\uparrow t) = (\uparrow t) \cap \text{Spec}(L)$ is $\Sigma(j)$ -exact. But, according to Proposition 2.5, if t is $\Sigma(j)$ -exact, then h_L^t is j -natural. This proves (a).

(b) The σ -exactness of $(\uparrow t) \cap \text{Spec}(L)$ is given by:

$$\sigma[(\uparrow t) \cap \text{Spec}(L)] = (\uparrow t) \cap \sigma(\text{Spec}(L)).$$

We now apply this to show that t is $\mathcal{J}(\sigma)$ -exact. To this end, assume that $t \leq x \in L$; then,

$$\begin{aligned} \mathcal{J}(\sigma)_L(x) &= \bigwedge \{ p \in \sigma(\text{Spec}(L)) : x \leq p \} \\ &= \bigwedge \{ p \in \sigma[(\uparrow t) \cap \text{Spec}(L)] : x \leq p \} = \mathcal{J}(\sigma)_{\uparrow t}(x); \end{aligned}$$

and the reader will note that it is in the second equality that σ -exactness is used. \blacksquare

Remark 3.9. Without the assumption that the nuclear typing j in Theorem 3.8 have spatial values one is able to “partially” obtain a converse to (a) in that theorem.

Assume that, for $t \in L$, $(\uparrow t) \cap \text{Spec}(L)$ is $\Sigma(j)$ -exact. This means that

$$\text{Spec}(jL) \cap \text{Spec}(\uparrow t) = \text{Spec}(j_{\uparrow t}(\uparrow t)).$$

Now observe that, if $j_L(x) = j_L(y)$ (with $t \leq x, y \in L$), then x and y determine the same closed set in $\text{Spec}(jL)$, namely $\{ p \in \text{Spec}(jL) : j_L(x) \leq p \}$, which trace on the same set in $\text{Spec}(j(\uparrow t))$; that is, $j_{\uparrow t}(x) = j_{\uparrow t}(y)$, proving that h_L^t is j -natural.

Unfortunately, this may not suffice to conclude that t is j -exact (see Example 5.16), and we are therefore unable to complete the argument for a converse to Theorem 3.8(a).

4 Algebraic Frames.

We specialize now to algebraic frames. The connection to the work of Escardó in [Es01] will become evident presently, but for now let us simply indicate that an algebraic frame is locally compact, in his sense. Without apology we will assume Zorn’s Lemma throughout, which makes every algebraic frame spatial.

Definition & Remarks 4.1. Let L be a complete lattice.

1. L is *algebraic* if and only if each $x \in L$ is a supremum of compact elements. $\mathfrak{k}(L)$ stands for the set of compact elements of L .
2. L is said to have the *finite intersection property* (abbr. *FIP*) if for any pair $a, b \in \mathfrak{k}(L)$ it follows that $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema. L is *coherent* if it is compact and has the FIP.
3. (See [MZ03, §4].) Suppose that L is an algebraic lattice, and j is a closure operator. Say that j is *inductive* if

$$j(x) = \bigvee \left\{ j(a) : a \in \mathfrak{k}(L), a \leq x \right\}.$$

It is easily seen that a closure operator is inductive if and only if it preserves joins of upward-directed sets; see [MZ03, Proposition 4.3] and also [Es01, §2]. Escardó calls an inductive nucleus *finitary* ([Es98]), and in [Es01] opts for the term *perfect* nucleus.

If j is inductive, then jL is algebraic and $\mathfrak{k}(jL) = j(\mathfrak{k}(L))$. If L is also a frame with the FIP and j is an inductive nucleus on L , then jL is an algebraic frame with the FIP.

4. (See [MZ03, §4].) Suppose that L is an algebraic frame with the FIP and that j is a nucleus on L . Let $\text{Ab}(j)$ stand for the set of all $x \in L$ such that $a \leq x$ (with a compact) implies that $j(a) \leq x$. Then $\text{Ab}(j)$ is an algebraic frame with the FIP. More precisely,

$$\widehat{j}(x) = \bigvee \left\{ j(a) : a \in \mathfrak{k}(L), a \leq x \right\}$$

defines an inductive nucleus such that $\widehat{j}L = \text{Ab}(j)$. \widehat{j} is the largest inductive closure operator below j . It is convenient to describe the passage $j \mapsto \widehat{j}$ as *inductivization*. Escardó ([Es98, Es01]) considers inductivization in a more general context.

5. Suppose that L is an algebraic frame. It is said to have *disjointification* if for each pair of compact elements $a, b \in L$, there exist disjoint $c \wedge d = 0$ in $\mathfrak{k}(L)$, such that $c \leq a$ and $d \leq b$, and $a \vee b = a \vee d = c \vee b$. Banaschewski ([Ba97]) calls this concept *coherent normality*.

If L has disjointification then, for each prime $p \in L$, $\uparrow p$ is a chain, and the converse is true assuming that L satisfies the FIP. This was first proved by Monteiro ([Mo54]). A proof may be found in [ST93, Lemma 2.1].

Next, we outline how to “extend” a nuclear typing j to its inductivization \widehat{j} (as a typing). We shall denote by $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$ the (non-full) subcategory of $\mathfrak{F}\mathfrak{rm}$ comprising all the algebraic frames with the FIP and all frame maps which carry compact elements to compact elements – called *coherent maps*.

Remarks 4.2. Let j be a nuclear typing on a subcategory \mathfrak{C} of $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$, and $h : A \longrightarrow B$ be a j -natural frame homomorphism. By first restricting to the compact elements of $\widehat{j}A$, and then extending to $\widehat{j}A$ itself, the existence of $J(h)$ in the diagram (2.1.1) implies that (4.2.1), below, also commutes.

$$(4.2.1) \quad \begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow j_A & & \downarrow j_B \\ JA & \xrightarrow{J(h)} & JB \\ \nearrow J(h)|_{\mathfrak{k}(A)} & & \nearrow \eta_B \\ j(\mathfrak{k}(A)) & \xrightarrow{\eta_A} & j(\mathfrak{k}(B)) \\ \searrow & & \searrow \\ \widehat{JA} & \xrightarrow{\widehat{j}(h)} & \widehat{JB} \end{array}$$

(In the diagram the unlabelled arrows are inclusions; η_A is the induced nucleus $\eta_A(x) = j_A(x)$, for $x \in \widehat{j}A$.) It should be evident that the \widehat{j} becomes a nuclear typing, and that h is also \widehat{j} -natural. We spell out the definition of $\widehat{J}(h)$; for each $x \in \widehat{j}A$,

$$\widehat{J}(h)(x) = \bigvee \left\{ J(h)(a) : a \in \mathfrak{k}(A), j_A(a) \leq x \right\}.$$

The next proposition tells for which j -natural frame quotients $\widehat{J}(h)$ remains surjective; we refer to such quotients as *j -quotients*.

Proposition 4.3. *Suppose the \mathfrak{C}_j -morphism $h : A \longrightarrow B$ is surjective. Then $\widehat{J}(h)$ is surjective if and only if the restriction $J(h)|_{\mathfrak{k}(A)}$ is surjective.*

Proof. If the restriction $J(h)|_{\mathfrak{k}(A)}$ is surjective, then by the definition of $\widehat{J}(h)$, coupled with the fact that the frames under consideration are algebraic, we have that $\widehat{J}(h)$ is surjective.

Conversely, suppose that $\widehat{J}(h)$ is surjective. If $b \in j_B(\mathfrak{k}(B))$, then $b = \widehat{J}(h)(x)$, for a suitable $x \in \widehat{JA}$. Then x is, in turn, expressible as a supremum $x = \bigvee_\lambda j_A(a_\lambda)$, with

each a_λ compact in A . Computing, $b = \vee_\lambda j_B(h(a_\lambda))$, and since b is compact in $\widehat{J}B$ and h is coherent, there are finitely many indices $\lambda_1, \lambda_2, \dots, \lambda_m$ such that

$$b = \vee_{i=1}^m j_B(h(a_{\lambda_i})) = j_B(h(\vee_{i=1}^m a_{\lambda_i})) = J(h)(j_A(a)),$$

where $a \equiv \vee_{i=1}^m a_{\lambda_i}$ is compact in A . \blacksquare

We return to the frame quotients h_A^t , and refer the reader to Proposition 2.5. It is easy to verify that h_A^t the compact elements of $\uparrow t$ are the ones of the form $a \vee t$, for all $a \in \mathfrak{k}(A)$, which implies that $\uparrow t$ is an $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$ -object and h_A^t is a coherent map.

Proposition 4.4. *Suppose that j is a nuclear typing on the subcategory \mathfrak{C} , and that h_A^t is a \mathfrak{C} -morphism. If $t \in A$ is j -exact then*

- (a) h_A^t is \widehat{j} -exact,
- (b) $\widehat{J}(h_A^t) = h_{\widehat{J}A}^{\widehat{j}A(t)}$, and
- (c) h_A^t is a j -quotient.

Proof. By Propositions 2.5 and 2.11, h_A^t is \widehat{j} -exact if and only if it is \widehat{j} -natural and (b) holds. So it is clear that (a) implies (b). Then (c) follows easily, since when h_A^t is j -natural it is also \widehat{j} -natural (4.2). Then one also realizes that (b) implies (a). Thus, it actually suffices to verify (b). But since $\widehat{J}(h_A^t)$ agrees with $h_{\widehat{J}A}^{\widehat{j}A(t)}$ on the compact elements of $\widehat{J}A$ – because $J(h_A^t) = h_{JA}^{jA(t)}$ – and $\widehat{J}A$ is algebraic, (b) follows. \blacksquare

Definition & Remarks 4.5. If $X \cong \text{Spec}(L)$, for some algebraic frame with the FIP L , then we call X an *algebraic space*, and denote the category of all algebraic spaces $\mathfrak{A}\mathfrak{S}\mathfrak{p}$. The reader will note that if X is an algebraic space, then (per our convention) $\mathfrak{D}(X) \cong L$, if L is an $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$ -object and $X = \text{Spec}(L)$. If j is a nuclear typing on the category \mathfrak{C} of $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$ -objects, then $\Sigma\mathfrak{C}$ consists of algebraic spaces.

We recall that the spectrum of an algebraic frame is elsewhere called a *spectral space*. X is spectral if and only if it is sober and has a base of compact open sets. Observe that X is algebraic if and only if it is spectral, with a base of compact open sets which forms a sublattice.

The following corollary follows from Theorem 3.8 and the preceding proposition.

Corollary 4.6. *Suppose that σ is a spatial selector on a category \mathfrak{A} of algebraic spaces; let $j = \mathcal{J}(\sigma)$. If σ is j -closed-hereditary (resp. closed-hereditary), then $\Sigma(\widehat{j})$ is j -closed-hereditary (resp. closed-hereditary).*

Here is the construction which is dual to inductivization of nuclei on an $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$ -object. Using it one should be able to prove the preceding corollary directly. We also present an account of the situation from the point of view of the patch topology on the spectrum.

Definition & Remarks 4.7. Let X be an algebraic space and express it as $X = \text{Spec}(L)$, for a suitable $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$ -object L . Let j be a nucleus on L .

(a) Dualizing the sequence of frame maps

$$L \xrightarrow{\hat{j}} \hat{j}L \xrightarrow{\eta} jL,$$

with η standing for the restriction of j to $\hat{j}L$, one obtains the inclusion of subspaces

$$\text{Spec}(jL) \subseteq \text{Spec}(\hat{j}L) \subseteq X,$$

in which the second inclusion satisfies the following: for each compact open set K of X , $K \cap \text{Spec}(\hat{j}L)$ is compact in $\text{Spec}(\hat{j}L)$. For obvious reasons we call a continuous map f between two topological spaces *coherent* if the inverse image of each compact open set is compact. Thus the inclusion of the subspace Y in X is coherent if and only if

$$K \text{ compact open in } X \implies K \cap Y \text{ compact open in } Y,$$

which is true, in turn, precisely when the dual frame quotient $\mathfrak{D}(X) \cong L \longrightarrow \mathfrak{D}(Y)$ is a coherent map. We shall also say that Y is *coherently embedded* in X .

(b) Consider now the *patch topology* on the algebraic space X : this is the family of open sets generated by the compact (hull-kernel) open sets and their complements. Viewing X as $\text{Spec}(L)$, as in (a), the basic open sets of the patch topology are the sets

$$\pi(a; b) \equiv \{ p \in \text{Spec}(L) : a \leq p, b \not\leq p \},$$

for all $a, b \in \mathfrak{k}(L)$.

Now, it can be shown that the sober subspace $S \subseteq X$ is *patch-closed* in X – that is, closed in the patch topology – if and only if the dual S^* is inductive. This is carried out in a more general setting in [Es01, Lemma 3.3], to which we refer the reader.

To outline a more-or-less direct proof, note, for the sufficiency, that if S^* is inductive, then $S = \text{Spec}(jL)$, where $j = S^*$. Then if p is prime and $p < j(p)$, there exists a compact $a \leq p$ such that $j(a) \not\leq p$. This uses the fact that j preserves joins of upward directed sets.

Conversely, assume S is patch-closed; it is easily shown that S is necessarily sober. The argument showing that S^* is inductive then proceeds in two stages. Note that the patch-closed sets are obtained by taking finite unions and arbitrary intersections of sets which are compact open or else closed in the spectral topology.

1. If S is closed, or compact open, then S^* is inductive.

Note that

$$S^*(U) = \begin{cases} U \cup (X \setminus S) & \text{if } S \text{ is closed,} \\ U \cap S & \text{if } S \text{ is compact open.} \end{cases}$$

In both cases it is clear that S^* is inductive.

2. The family \mathcal{F} of sober sets S for which S^* is inductive is closed under all intersections and finite unions. (See [Es98, Lemma 3.1.8]; the argument we outline below is substantially that of Escardó.)

Suppose $A, B \in \mathcal{F}$; $(A \cup B)^* = A^* \wedge B^*$, the latter infimum being taken pointwise. Since an inductive nucleus preserves upward directed joins, one is able to show that $(A \cup B)^*$ is inductive.

Now suppose $S_i \in \mathcal{F}$ ($i \in I$). Since each S_i is sober, we have $(\cap_i S_i)^* = \vee_i S_i^*$. Let \mathcal{S} stand for the set of all finite compositions of the S_i^* s. Once more we appeal to the observation that a nucleus is inductive if and only if it preserves joins of upward directed sets. The trick is to show that $\vee_i S_i^* = \bigvee \mathcal{S}$; it should be clear that the left side exceeds the right. It is a fairly routine operation to show that $\bigvee \mathcal{S}$ is a nucleus and that it is inductive.

A simple application of the duality between spatial frames and sober spaces, restricted to $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$ and $\mathfrak{A}\mathfrak{S}\mathfrak{p}$, together with 4.1.4 and the comments in 4.7(b) then yield the following.

Proposition 4.8. *Let \mathfrak{A} be a subcategory of algebraic spaces and σ be a spatial selector; denote $j = \mathcal{J}(\sigma)$. For each \mathfrak{A} -object X ,*

$$\widehat{\sigma}(X) \equiv \text{Spec}(\widehat{\mathcal{J}\mathfrak{D}}(X))$$

is coherently embedded in X and contains $\sigma(X)$. Moreover, $\widehat{\sigma}(X)$ is the least coherently embedded subspace of X that contains $\sigma(X)$. It is also the patch-closure of $\sigma(X)$ in X .

Proof. The reader is reminded that $\mathcal{J}\mathfrak{D}(X)$ is canonically isomorphic to $\mathfrak{D}(\sigma(X))$, by Corollary 3.4, and that $\text{Spec}(\mathfrak{D}(\sigma(X)))$ is the sober closure of $\sigma(X)$ (3.1.2b). The details are left as an exercise. ■

Let σ be a spatial selector. In keeping with the language of [Es01], we shall refer to the selector $\widehat{\sigma}$ introduced in Proposition 4.8 as the *patch* of σ .

5 Applications: Hausdorff Typings and Selectors.

As was indicated in the introduction to this paper, the study of nuclear typings and spatial selectors was generated by our ongoing work on dimension in algebraic frames, and, indeed, an understanding of the correspondences described above seems like a prerequisite to a proper discussion of dimension theories, which is being prepared in [MZ04 ∞]. Many of the natural settings that come up in the work on dimension involve categories of normal algebraic frames with the FIP, and from the point of view of spatial selectors we arrive, dually, at the question below, which we take here as motivation to present a number of examples and illustrations.

Question 5.1. *When is a spatial selector the patch of a Hausdorff selector?*

There is a subtext to this question: in view of our preoccupation over dimension, what we would really like to know is when a spatial selector is the patch of a closed-hereditary Hausdorff selector.

At this point we need to recollect the relationship between normality in algebraic frames and Hausdorff subspaces of their spectra. The reader is referred to 1.1.6. Please note the obvious; that a space X satisfies the normal separation axiom if and only if $\mathfrak{O}(X)$ is a normal frame.

Remarks 5.2. It is well known that in a normal frame L , $\text{Max}(L)$ is a Hausdorff subspace, and that L is a *pm-frame*; that is, each prime is exceeded by at most one maximal element. If L is coherent and a *pm-frame* then it is also normal. Recall that the $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$ -object L has disjointification if and only if $\downarrow c$ is normal, for each $c \in \mathfrak{k}(L)$ (4.1.5).

Regarding the three following conditions, it is also well known that, in any $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$ -object L and for any subset S of $\text{Spec}(L)$, (a) \Rightarrow (b) \Leftrightarrow (c), and if L has disjointification, (c) implies (a).

- (a) S is Hausdorff.
- (b) S is T_1 .
- (c) S is trivially ordered.

Finally, observe that any Hausdorff space is sober.

Remark 5.3. Suppose that j is a nuclear typing on the subcategory \mathfrak{C} of $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$. We say that j is a *Hausdorff typing* if there is a Hausdorff selector σ (defined on $\Sigma\mathfrak{C}$) such that $\mathcal{J}(\sigma) = j$. Per Proposition 3.7(b), if this is the case, then σ is unique and $\Sigma(j) = \sigma$.

An algebraic formulation of Question 5.1 is then: *when is an inductive nucleus the inductivization of a Hausdorff typing?*

Among the Hausdorff selectors we distinguish those defined on a category of normal algebraic spaces, having values which are maximal in the spectral ordering (5.2). A selector whose values are maximal will be called a *max-selector*, and the nuclear typing it induces a *max-typing*.

With the question posed in 5.1 in mind, we now consider two examples which are prominent in our recent work in [MZ03, M04a, MZ04a, M04b, MZ04b]. We briefly review the pertinent definitions, referring the reader to [MZ03, §5 & §6] and [M73] for additional background. In [MZ04b] there is a further development of the frame of z -elements, in the context of Yosida frames.

Example 5.4. Let L be an algebraic frame with the FIP. First, d -typing will refer to the inductivization of the polar typing p . Explicitly, for each $x \in L$,

$$d(x) = \bigvee \left\{ p(c) : c \leq x, c \in \mathfrak{k}(L) \right\}.$$

d is the largest inductive nucleus fixing all polars. Now, each polar is a meet of minimal primes, and every minimal prime is a d -element ([MZ03, Lemma 2.2]). This makes it easy to see that d is also the inductivization of $\mathcal{J}(\text{Min})$. By [MZ03, Lemma 2.2], Min is a Hausdorff selector on $\mathfrak{A}\mathfrak{S}\mathfrak{p}$, whose values are zero-dimensional spaces.

Note that if O is open in $\text{Spec}(L)$, then $O = \text{coz}(x)$ for some $x \in L$, which is canonically homeomorphic to $\text{Spec}(\downarrow x)$ (via the map $q \mapsto q \wedge x$, per [MZ03, Lemma 4.6]). Under this homeomorphism $O \cap \text{Min}(L)$ maps onto $\text{Min}(\downarrow x)$ ([MZ03, Corollary 4.7]). This suffices to establish that Min is open-hereditary.

It is easy to see that Min is not always closed-hereditary on $\mathfrak{A}\mathfrak{S}\mathfrak{p}$. Thus, with regard to 5.1, we see that $\Sigma(d) = \widehat{\text{Min}}$, i.e., it is the patch of a Hausdorff selector, although not necessarily of a closed-hereditary Hausdorff selector.

Example 5.5. Let L be an algebraic frame with the FIP. We say that $m < x$ is *maximal under x* if m is maximal in $\downarrow x$. L is an *archimedean lattice* if, for each $c \in \mathfrak{k}(L)$, $\bigwedge \text{Max}(\downarrow c) = 0$. This concept first appeared in [M73]. Next, $x \in L$ is *upper-archimedean* if $\uparrow x$ is archimedean. Denote the set of all upper-archimedean elements of L by $\mathfrak{a}^\uparrow(L)$. Note that if L is compact then $x \in \mathfrak{a}^\uparrow(L)$ precisely when x is an infimum of maximal elements of L .

By [MZ03, Lemma 6.2], $\mathfrak{a}^\uparrow(L)$ is nuclear, and so there is a nucleus ar on L such that $arL = \mathfrak{a}^\uparrow(L)$. Note that ar is dense if and only if L is archimedean. This defines a nuclear typing, here referred to as the *archimedean typing*.

We also consider the inductivization, $z = \widehat{ar}$, called z -typing. Explicitly,

$$z(x) = \bigvee \left\{ ar(c) : c \leq x, c \in \mathfrak{k}(L) \right\}.$$

Observe that $zL = \text{Ab}(ar)$ (4.1.4).

There is one reasonable sufficient condition for a spatial selector to be the patch of a closed-hereditary selector with values which are antichains. We doubt that the condition in Proposition 5.7 is necessary, but we can offer no counterexamples. The proposition is prefaced by a brief account of Yosida frames.

Remark 5.6. Suppose that L is an algebraic frame. L is a *Yosida frame* if every compact element is a meet of maximal elements. If L is a Yosida frame, then $\text{Max}(L)$ is dense and L is archimedean; moreover, $L = zL$, and the converse of this is true if L is compact ([MZ04b, Proposition 2.11]). Regular algebraic frames are Yosida, but not conversely; examples are given in [MZ04b].

For any nuclear typing j , we put $\text{Max}_j(L) \equiv \text{Max}(jL)$; below we view Max_j as a spatial selector.

Proposition 5.7. *Suppose that j is an inductive typing defined on the subcategory \mathfrak{C} of $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$, and that for each \mathfrak{C} -object L , jL is a Yosida frame. Then $j = \mathcal{J}(\widehat{\text{Max}_j})$. Further,*

- (a) *if every $t \in L$ is j -exact then Max_j is closed-hereditary;*
- (b) *if, in addition, \mathfrak{C} consists of frames with disjointification, then Max_j is a Hausdorff selector, and j induces the patch of a Hausdorff selector.*

Proof. Suppose that L is a \mathfrak{C} -object; as jL is a Yosida frame, each j -element is a supremum of meets of members of $\mathcal{J}(\text{Max}_j)(L)$. Since each such supremum is necessarily in jL , the claim follows.

Now (a): we have that $\Sigma(j) = \widehat{\text{Max}_j}$, and, by assumption, $\text{Spec}(jL) \cap (\uparrow t) = \text{Spec}(j(\uparrow t))$, for each $t \in L$. Evidently then also, $\text{Max}_j(L) \cap (\uparrow t) = \text{Max}_j(\uparrow t)$.

(b) is clear from the comments in 5.2. ■

It should be evident by now, that if Question 5.1 is to have an affirmative answer for a given inductive typing j , some significant restrictions will have to be placed on its values, as well as on the category over which it is defined. There are numerous examples which are both interesting and illuminating in this regard; we shall take the opportunity to discuss them in some detail. We find 5.13 and 5.15 especially instructive.

Example 5.8. *Max* as Hausdorff typing.*

We consider the Max typing of Example 2.3. Note that $\text{Max}^* = \mathcal{J}(\text{Max})$ and that Max is closed-hereditary (Example 3.5). It is a Hausdorff typing as long as it is defined on normal frames (5.2). In fact, according to Proposition 3.7(b), $\text{Max} = \Sigma(\mathcal{J}(\text{Max}))$ if and only if the values of Max are sober spaces. We have already seen that if L is the frame of semiprime ideals of the ring of integers and X is its spectrum, then $\Sigma(\mathcal{J}(\text{Max}))(X) = X$ (and *not* $\text{Max}(L)$). X is not sober, and L is not a *pm*-frame.

Example 5.9. *When an inductive typing is Hausdorff.*

Suppose that j is an inductive nuclear typing. If j is Hausdorff then there is a Hausdorff selector σ such that $j = \mathcal{J}(\sigma)$. But then $\sigma = \Sigma(j)$, so that for each \mathfrak{C} -object L , $\sigma(\text{Spec}(L)) = \text{Spec}(jL) = jL \cap \text{Spec}(L)$, which is a Hausdorff space. This implies that jL is regular; (see [MZ03, Theorem 2.4].) The converse, that regularity of jL implies that $\text{Spec}(jL)$ is Hausdorff, is also clear.

Thus, if \mathfrak{C} is a subcategory of $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$ -objects L for which jL is regular, then, apart from j being a Hausdorff typing, its induced (or inducing) selector $\Sigma(j)$ picks the maximal j -elements. This is a max-selector precisely when (for each \mathfrak{C} -object L) every maximal j -element of L is maximal. For a discussion of frames with these features the reader is referred to [M04b, §5], and, in particular, [M04b, Proposition 5.11].

Example 5.10. *When the polar typing is Hausdorff.*

(See also Examples 3.6 and 5.13.) Consider $x \mapsto p(x) = x^{\perp\perp}$ on $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$. For any frame L (whatsoever), we note that the polar $x \in L$ is prime precisely when it is of the form $x = b^{\perp}$, such that $\downarrow b$ is a chain; (see [M73, Lemma 2.1]). In [M73] an element $b \in L$ with the feature that $\downarrow b$ is a chain is called *basic*. Note that $b > 0$ is basic if and only if $b^{\perp\perp}$ is an atom of pL . Therefore, if we choose L to be an algebraic frame without basic elements, then $\Sigma(p)(\text{Spec}(L))$ is empty, and $\mathcal{J}(\Sigma(p))(\text{Spec}(L))$ is the trivial frame.

Thus, the values of p are not, in general, spatial. Nonetheless, for each algebraic space $\text{Spec}(L)$, define $\tau(\text{Spec}(L))$ to be the set

$$\{p \in \text{Min}(L) : p = b^{\perp} \text{ for suitable basic } b > 0\}.$$

Using [MZ03, Lemma 2.2] yet again, shows that $\tau(\text{Spec}(L))$ is none other than the set of isolated points of $\text{Spec}(L)$. (We shall consider the selector of closed isolated points in Example 5.13.)

Now, if $p = \mathcal{J}(\sigma)$, for some Hausdorff selector σ (on a suitable subcategory \mathfrak{A} of algebraic spaces), then $\sigma = \Sigma(p)$, whence

$$\sigma(\text{Spec}(L)) = \text{Spec}(L) \cap pL \subseteq \tau(\text{Spec}(L));$$

this implies that $\mathcal{J}(\tau) \leq p$, on the category \mathfrak{C} of duals of \mathfrak{A} . As the reverse inequality is obvious, we see that $\mathcal{J}(\tau) = p$ is forced. Applying Proposition 3.7(b), it follows that $\tau = \sigma$. If this is the case then $p = \mathcal{J}(\tau)$, which forces a restriction to the subcategory $\mathfrak{B}\mathfrak{a}$ of $\mathfrak{A}\mathfrak{F}\mathfrak{r}\mathfrak{m}$ -objects L in which every polar is the meet of polars of the form b^{\perp} , with $b > 0$ basic.

It is easy to see that L has this feature precisely when it *has a basis*, i.e., each $0 < x \in L$ exceeds some basic b . Equivalently, L has a basis if and only if pL is an *atomic* boolean algebra; that is, each polar of L is the supremum (in pL) of atoms. The reader is referred to [M73, Lemma 2.1 & Theorem 2.2] for further details.

Dually, one should restrict τ to all algebraic spaces having a dense set of isolated points.

Example 5.11. *When the archimedean typing is Hausdorff.*

Suppose that \mathfrak{C} is a category of $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$ -objects on which $ar = \mathcal{J}(\alpha)$, for a suitable Hausdorff selector α . As in other cases, $\alpha = \Sigma(ar)$, which implies that $\alpha(\text{Spec}(L)) = arL \cap \text{Spec}(L)$. Next, notice that any prime p for which $\uparrow p$ is archimedean is, in fact, maximal – and conversely. Thus, $\alpha(\text{Spec}(L)) = \text{Max}(L)$, and we have that $ar = \mathcal{J}(\alpha) = \text{Max}^*$.

Thus, \mathfrak{C} consists of objects in which every proper upper-archimedean element is a meet of maximal elements. In fact, if ar is induced by any sober selector α whatsoever, then, necessarily, $\alpha = \text{Max}$, and the latter is Hausdorff if restricted as in Example 5.8.

This situation has received some attention in the literature, in the context of ℓ -ideals of Φ -algebras. In rings of continuous functions $C(X)$, every upper-archimedean element of the frame of semiprime ideals is a meet of maximal elements precisely when the underlying space X is Lindelöf ([NP72, Corollary 4.2]). This is generalized to Φ -algebras in [P71, 3.3]; see also [H76].

Per a personal communication with the authors, A. W. Hager has extended the above characterizations to the frame $\ker(\mathbf{W})$ of \mathbf{W} -kernels of the category \mathbf{W} of all archimedean lattice-ordered groups with a designated weak unit, together with the ℓ -homomorphisms that preserve the designated unit. Now $\ker(\mathbf{W})$ may be obtained as the set of upper-archimedean elements of a suitable algebraic frame with the FIP. It is well known ([BH91, §3.2]) that $\ker(\mathbf{W})$ is a regular frame which is Lindelöf, in the sense that $1 = \bigvee S$ implies that $1 = \bigvee T$, for a suitable countable subset T of S . In terms of the Yosida space YG of a \mathbf{W} -object G and its Yosida representation, it then easily follows that if each \mathbf{W} -kernel of G is an intersection of real \mathbf{W} -kernels then the subspace of YG of all real \mathbf{W} -kernels is Lindelöf.

The next two examples ought to be viewed in light of the characterization of Proposition 3.3. To put Example 5.13 in proper perspective, the following remark might be useful to the reader.

Remark 5.12. Let X be a topological space. Recall that an open set U of X is said to be *regular open* if $U = \text{int}_X \text{cl}_X U$, or, equivalently, if it is the interior of a closed set. Relative to the frame $\mathfrak{D}(X)$ of all open sets, an open set is regular open if and only if it is a polar. This is well known, and easy to verify directly. Viewed topologically then, the typing $U \mapsto \text{int}_X \text{cl}_X U$, on $\mathfrak{D}(X)$ is none but the polar typing p . In effect then, Example 5.13 below shows that, in general, an element of an $\mathfrak{A}\mathfrak{F}\mathfrak{rm}$ -object need not be p -exact, as announced in §2. (Recall Proposition 2.6, stating that every polar is p -exact.)

Now, when X is also algebraic, and j is a nuclear typing defined on a category \mathfrak{C} including the object $\mathfrak{D}(X)$, observe that $\mathfrak{D}(X)$ is j -semisimple if and only if $jU \subseteq \text{int}_X \text{cl}_X U$, for each open set U of X .

Example 5.13. *A max- (and Hausdorff) selector, Is , which is $\mathcal{J}(\sigma)$ -closed-hereditary, but not closed-hereditary.*

For any space X , let $\text{Is}(X)$ denote the set of isolated closed points of X . Observe right away that Is is a discrete selector, and therefore Hausdorff.

Now let us see that Is is not closed-hereditary. Let $\alpha\mathbb{N}$ denote the one-point compactification of the discrete space \mathbb{N} . Note that $\text{Is}(\alpha\mathbb{N}) = \mathbb{N}$. Let $L = \mathfrak{D}(\alpha\mathbb{N})$; then L is a regular algebraic frame, and $\alpha\mathbb{N} \cong \text{Spec}(L) = \text{Max}(L)$. Next, let q be the single point of $\alpha\mathbb{N}$ which is not isolated. We have that $\text{Is}(\{q\}) = \{q\}$, whereas $\{q\} \cap \text{Is}(\alpha\mathbb{N}) = \emptyset$, proving that Is is not closed-hereditary.

However, Is is $\mathcal{J}(\text{Is})$ -closed-hereditary, by application of Proposition 3.3, as follows. Now, according to that result,

$$\mathcal{J}(\text{Is})(U) = \text{int}_X[(X \setminus \text{Is}(X)) \cup U],$$

for each $U \in \mathfrak{D}(X)$, with X sober. Assuming, further, that $\mathfrak{D}(X)$ is $\mathcal{J}(\text{Is})$ -semisimple – that is, that X has a dense subset of closed isolated points, as is the case for $\alpha\mathbb{N}$ – then it is an easy exercise to show that $\mathcal{J}(\text{Is})(U) = \text{int}_X \text{cl}_X U$. Further, $U \in \mathfrak{D}(X)$ is $\mathcal{J}(\text{Is})$ -closed precisely when U is regular open. According to Theorem 3.8(b), it must be settled that $\text{Is}(S) = \text{Is}(X) \cap S$, for each regular closed set S . This is obvious.

Prior to the next example, we review a basic topological concept. The main reference is [GJ76, Theorem 14.29].

Definition & Remarks 5.14. (a) Let X be a topological space and $p \in X$. Call p a *P-point* if for any sequence $U_1, U_2, \dots, U_n, \dots$ of neighborhoods of p , $\bigcap_n U_n$ is a neighborhood of p . This is perhaps a slightly unconventional way to define this concept, but it agrees with the one in [GJ76, 4L] for Tychonoff spaces. For non-Hausdorff spaces we believe the present one to be the most convenient formulation.

A *P-space* is one in which every point is a *P-point*. Let $\sigma_P(X)$ denote the collection of all closed *P-points*. It should be clear that $\sigma_P(X)$ is a *P-space*. Note that $\sigma_P(X)$ is not necessarily the largest subspace of X which is a *P-space*. On the other hand, every Tychonoff *P-space* is zero-dimensional.

(b) To better understand Example 5.15, we also include the following review. First, we shall consider a totally ordered space (X, \leq) below, always relative to the interval topology. The reader is urged to have a look at [GJ76, 3O]: such spaces are normal (Hausdorff) spaces ([GJ76, 3O.9]), and compact if and only if order complete ([GJ76, 3O.7]).

We shall use the ordinal line (W, \leq) of all countable ordinals and its order completion (and compactification) (W^*, \leq) – thereby also employing the notation of [GJ76, Chapter 5]. The largest element of W^* is the ordinal ω_1 , and this is the only nonisolated *P-point* of the space.

Moreover, note the following ([GJ76, 5O.1]); let (X, \leq) be a totally ordered space. Then $p \in X$ is a *P-point* if and only if it is neither the limit of an increasing sequence nor the limit of a decreasing sequence.

Example 5.15. A max-selector, σ_P , such that not every $\mathcal{J}(\sigma_P)$ -element is $\mathcal{J}(\sigma_P)$ -exact.

Consider σ_P as a spatial selector on the subcategory of $\mathfrak{A}\mathfrak{S}\mathfrak{p}$ consisting of all normal algebraic spaces. That makes σ_P a max, Hausdorff selector.

Let $X = W^* \cup \{1/n : n \in \mathbb{N}\}$, with $U = \{1/n : n \in \mathbb{N}\}$ ordered as a subset of the rationals, such that $x < 1/n$, for each $x \in W^*$ and each $n \in \mathbb{N}$. Notice that the largest element ω_1 of W^* , while a P -point of W^* , is a non- P -point of X ; (see 5.14(b)). We have that (X, \leq) is a compact totally ordered space which is zero-dimensional, and therefore an algebraic space. We now show that, with $j = \mathcal{J}(\sigma_P)$, U is a j -element of $\mathfrak{D}(X)$ which is not j -exact.

First, U is open; it is also easily verified that

$$U = \text{int}_X[(X \setminus \sigma_P(X)) \cup U] = jU.$$

Observe as well that $U \subseteq \sigma_P(X)$. It suffices then to show that $W^* = X \setminus U$ (as a closed set in X) is not σ_P -exact. As we have already noted, $\omega_1 \in \sigma_P(W^*)$. However, $\omega_1 \notin W^* \cap \sigma_P(X)$, as ω_1 is not a P -point of X .

Finally, an example which shows that, in Proposition 2.5, the fact that h_A^t is j -natural need not imply that $t \in A$ is j -exact.

Example 5.16. The spatial selector is the complement of Is in Example 5.13; that is, for any space X , $\sigma(X) = X \setminus \text{Is}(X)$. As $\text{Is}(X) \cap Y \subseteq \text{Is}(Y)$, for any subspace Y of X , we have that $Y \subseteq X$ implies that $\sigma(Y) \subseteq \sigma(X)$. Abbreviate $j = \mathcal{J}(\sigma)$.

Now, by Proposition 3.3, if $U \in \mathfrak{D}(X)$, then

$$j_{\mathfrak{D}(X)}(U) = \text{Is}(X) \cup U,$$

and thus, U is a j -element if and only if it contains all the isolated closed points of X . On the other hand, for each open U in X , $\uparrow U$ is to be identified with $\mathfrak{D}(X \setminus U)$ via the map $V \mapsto V \setminus U$, for each open set V containing U . With this identification it is easy to check that $j_{\uparrow U}$ is the map $V \mapsto \text{Is}(X \setminus U) \cup V$ (for V open and containing U).

Now choose $U \in J\mathfrak{D}(X)$; if V, W are open sets containing U , then both must contain $\text{Is}(X)$, so that if $j_{\mathfrak{D}(X)}(V) = j_{\mathfrak{D}(X)}(W)$, we actually have $V = W$. This makes it obvious that $h_{\mathfrak{D}(X)}^U$ – and, indeed, any continuous map out of X – is j -natural.

Yet a j -element U is not j -exact, unless $X \setminus U$ is σ -exact. The same illustration of Example 5.13, with $X = \alpha\mathbb{N}$ and $U = \mathbb{N}$, demonstrates that the closed set $\{q\}$ is not σ -exact, as $\sigma(\{q\}) = \emptyset$, whereas $\{q\} \cap \sigma(\alpha\mathbb{N}) = \{q\}$.

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