

Sublattices Generated by Polars

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ABSTRACT. In an algebraic frame L the complete sublattice $CP(L)$ generated by the polars of L is studied, in comparison with $FP(L)$, the subframe generated by the polars. It is shown, by an example from the theory of ℓ -groups, that these are distinct, in general. The relationship between $FP(L)$, $CP(L)$, and other established constructs, closely related to the boolean algebra of polars, is also studied.

This investigation into sublattices generated by the polars of a frame began as an effort to generalize to frames the ideas contained in [XZ97] for lattice-ordered groups. It has evolved into a study of the subframe of regular elements of an algebraic frame, and, more specifically, of the question of when this subframe coincides with the regular coreflection of an algebraic frame. This paper deals with the polars of an algebraic frame, *per sé*, and with various sublattices that are closely related to the polars. Issues having to do with regularity are discussed in [MZ06b].

Let G be a lattice-ordered group. In [XZ97] is introduced the complete sublattice $\mathcal{C}_c(G)$ of the frame of all convex ℓ -subgroups of G , denoted $\mathcal{C}(G)$, generated by the polars of G . This clearly includes all the joins of polars, but, as we will show, not every member of $\mathcal{C}_c(G)$ is a join of polars. Let $\mathcal{P}(G)$ denote the boolean algebra of polars of G , and $\mathcal{C}_f(G)$ denote the subframe of $\mathcal{C}(G)$ generated by $\mathcal{P}(G)$. In general, $\mathcal{C}_c(G) \neq \mathcal{C}_f(G)$, and we give an example (in §2) showing that $\mathcal{C}_f(G)$ is, in general, not closed under set-theoretic intersection. It is an open question whether $\mathcal{C}_f(G)$ is an algebraic lattice.

We generalize the constructs described in the preceding paragraph to algebraic frames, although the intuition guiding these ideas comes squarely from lattice-ordered groups. The first section reviews the necessary background in frame theory. It also compares the new constructs to some closely related sublattices recently studied in the literature (e.g., [MZ03]). The distinction between the various sublattices introduced in §1 is left for §4 and §5; a number of examples is given in the latter, for rings of continuous functions. In §3 the branching of primes is related to suprema of polars. The concluding section lists the main open questions.

1 Frame-theoretic Preliminaries.

The commentary below is a catalogue of background material on frames and algebraic frames, in particular. We refer the reader to [J82] and [JT84] for general background on frames, and to [MZ03] for additional material on closure operators.

Definition & Remarks 1.1. Throughout, L is a complete lattice. The top and bottom are denoted 1 and 0 , respectively. For $x \in L$, $\uparrow x$ (resp. $\downarrow x$) stands for the set of elements $\geq x$ (resp. $\leq x$).

1. $c \in L$ is *compact*: $c \leq \bigvee_{i \in I} x_i$ implies that $c \leq \bigvee_{i \in F} x_i$, for a suitable finite subset F of I . L is *compact* if 1 is compact.

L is *algebraic*: each $x \in L$ is a supremum of compact elements. $\mathfrak{k}(L)$ stands for the set of compact elements of L .

2. L has the *finite intersection property* (abbr. *FIP*): $a, b \in \mathfrak{k}(L)$ implies that $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema.
3. L is *coherent*: it is compact and has the FIP.
4. L is a *frame* if the following distributive law holds: for any $S \subseteq L$,

$$a \wedge \left(\bigvee S \right) = \bigvee \left\{ a \wedge s : s \in S \right\}.$$

It is well known that an algebraic lattice is a frame as long as it is distributive.

5. $p \in L$ is *prime*: $p < 1$ and $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$.

$\text{Spec}(L)$ shall denote the set of prime elements of L . When we have occasion to consider a topology on $\text{Spec}(L)$ and its subsets (such as in Proposition 4.1), the topology in question will be the hull-kernel topology.

6. Let L be a frame. For each $a \in L$, let

$$a^\perp = \bigvee \{ x \in L : x \wedge a = 0 \}.$$

$x \in L$ is a *polar*: $x = y^\perp$, for some $y \in L$. It is well known that the set $P(L)$ of all polars forms a complete boolean algebra, in which infima agree with those in L .

7. Let L be a frame. $a \preceq b$: $b \vee a^\perp = 1$.

$x \in L$ is *regular* if

$$x = \bigvee \left\{ a \in L : a \preceq x \right\}.$$

L is *regular* if each element of L is regular.

8. (See [JT84].) Let L be a frame and suppose that $j : L \longrightarrow L$ is a closure operator; jL designates $\{x \in L : j(x) = x\}$; its members are called *j-elements*.

j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$.

9. (See [MZ03, §4].) Suppose that L is an algebraic lattice, and j is a closure operator. Say that j is *inductive* if

$$j(x) = \bigvee \{ j(a) : a \in \mathfrak{k}(L), a \leq x \}.$$

Then jL is algebraic and $\mathfrak{k}(jL) = j(\mathfrak{k}(L))$. If L is also a frame and j is a nucleus on L , then jL is an algebraic frame as well.

Observe, in addition, that if L is an algebraic frame and j is an inductive nucleus on L , then (a) $\text{Spec}(jL) = \text{Spec}(L) \cap jL$, and (b) if L has the FIP then so does jL .

10. (See [MZ03, §4].) Suppose that L is an algebraic frame with the FIP and that j is a nucleus on L . Let $\text{Ab}(j)$ stand for the set of all $x \in L$ such that $a \leq x$ (with a compact) implies that $j(a) \leq x$. Then $\text{Ab}(j)$ is an algebraic frame with the FIP. More precisely,

$$\widehat{j}(x) = \bigvee \{ j(a) : a \in \mathfrak{k}(L), a \leq x \}$$

defines an inductive nucleus such that $\widehat{j}L = \text{Ab}(j)$.

11. Closure operators on L are partially ordered by defining $j_1 \leq j_2$ if $j_1(x) \leq j_2(x)$ for each $x \in L$, which, in turn, is equivalent to $j_2L \subseteq j_1L$. Under these stipulations, and using the notation of 10, \widehat{j} is the largest inductive closure operator below j . The passage $j \mapsto \widehat{j}$ is referred to as *inductivization*.

Remark 1.2. It is worth underscoring that we shall assume and liberally apply Zorn's Lemma, which guarantees that all algebraic frames are *spatial*, that is, every element is a meet of primes.

Definition & Remarks 1.3. Suppose that L is a frame.

Let $\text{FP}(L)$ denote the subframe of L generated by $P(L)$. In view of the frame law, $\text{FP}(L)$ consists of the set of all joins of polars of L .

In between $\text{FP}(L)$ and L we have $\text{CP}(L)$, the complete sublattice generated by $P(L)$. The example in §2 shows that $\text{FP}(L)$ is, in general, not closed under (even countable) meets of L . In particular, $\text{FP}(L) \neq \text{CP}(L)$. Notice, however, that if $\text{FP}(L)$ is closed under meets of L , then $\text{FP}(L) = \text{CP}(L)$; the converse is obvious.

The following is well known ([Bi67, p. 130]).

Lemma 1.4. *For any frame L , the following are equivalent:*

- (a) $P(L)$ is a sublattice of L .
- (b) Each polar is complemented.
- (c) $a^\perp \vee b^\perp = (a \wedge b)^\perp$ holds in L .

Definition & Remarks 1.5. Suppose now that L is an algebraic frame with the FIP. $\text{Min}(L)$ denotes the set of all minimal primes of L . Let $\text{Min}^*(L)$ denote the collection of all meets of minimal primes of L .

Next, we have the inductive nucleus d on L defined by $d = \widehat{(\cdot)^{\perp\perp}}$. The members of dL are the d -elements; dL is an algebraic frame with the FIP as well, and, in the notation of 1.1.10, we have that

$$dL = \{ x \in L : c \leq x, c \in \mathfrak{k}(L) \implies c^{\perp\perp} \leq x \}.$$

(See [MZ03, §5] for a detailed discussion of d -elements. The concept is motivated by the d -ideals of [HuP80a, HuP80b].)

We now collect some basic information about the constructs assembled thus far.

Proposition 1.6. *Suppose that L is an algebraic frame with the FIP. Then*

- (a) $\text{CP}(L)$ is the complete sublattice generated by $\text{Min}(L)$.
- (b)
- (†) $P(L) \subseteq \text{Min}^*(L) \subseteq dL \subseteq \text{FP}(L) \subseteq \text{CP}(L)$.
- (c) If $dL = \text{FP}(L)$, then also $\text{FP}(L) = \text{CP}(L)$.
- (d) If $P(L)$ is a sublattice of L , then $dL = \text{FP}(L) = \text{CP}(L)$.

Proof. (a) It is enough to observe the well known fact that each $p \in \text{Min}(L)$ is a join of polars of the form c^\perp , with $c \not\leq p$, and $c \in \mathfrak{k}(L)$ ([MZ03, Lemma 2.2]), and that each polar is an infimum of minimal primes.

(b) We have just observed that $P(L) \subseteq \text{Min}^*(L)$. On the other hand, since each minimal prime is an upward directed supremum of polars, it is also a d -element, and we have that $\text{Min}^*(L) \subseteq dL$, because dL is closed under meets.

(c) If $dL = \text{FP}(L)$, then the latter is closed under all meets, and, as already observed in 1.3, $dL = \text{FP}(L) = \text{CP}(L)$.

(d) Since $P(L)$ is a sublattice of L , any join of polars $\bigvee S$ (with $S \subseteq P(L)$) may be rewritten

$$\bigvee S = \bigvee \left\{ \bigvee F : F \subseteq S, F \text{ finite} \right\},$$

and the finite subsets of S form an upward directed system. Therefore, since each $\bigvee F \in P(L)$, and $P(L) \subseteq dL$, we have that $\bigvee S \in dL$. This shows that $\text{FP}(L) = dL$, and by (c), $\text{FP}(L) = \text{CP}(L)$. ■

Now, while it is unknown whether $\text{FP}(L)$ is in general algebraic, $\text{CP}(L)$ is always. The proof is modeled after one supplied by X. Zheng for frames of convex ℓ -subgroups of a lattice-ordered group.

Proposition 1.7. *Suppose that L is any algebraic frame. Then $\text{CP}(L)$ is algebraic. More precisely, the map $x \mapsto c(x)$, which assigns each $x \in L$ to the least member $c(x)$ of $\text{CP}(L)$ exceeding x is an inductive closure operator, and $cL = \text{CP}(L)$.*

Proof. Observe at the outset that the map c is clearly a closure operator with $cL = \text{CP}(L)$, and in view of the fact that $\text{CP}(L)$ is a complete sublattice of L , c preserves arbitrary suprema. This implies that each $c(a)$, with a compact in L , is compact in $\text{CP}(L)$. For assume that $c(a) \leq \bigvee S$, for some subset S of $\text{CP}(L)$. Then $a \leq \bigvee S$, and so there is a finite subset $T \subseteq S$ such that $a \leq \bigvee T$. Applying c to both sides yields that $c(a) \leq \bigvee T$, proving the claim. Then it is also clear that $\text{CP}(L)$ is algebraic. ■

Conspicuous for its absence in Proposition 1.7 is the claim that the operator c is a nucleus. In general we do not know this to be true; it is true in frames with disjointification. We review this concept next.

Definition & Remarks 1.8. Let L be an algebraic frame. We say that L has *disjointification* if for each pair of compact elements $a, b \in L$ there exist disjoint $c, d \in \mathfrak{k}(L)$ such that

1. $c \leq a$ and $d \leq b$, and
2. $a \vee b = a \vee d = c \vee b$.

Observe that if L has disjointification and, given $a, b \in \mathfrak{k}(L)$, $c, d \in \mathfrak{k}(L)$ are chosen to witness the disjointification of a and b , then $a = (a \wedge b) \vee c$ and $b = (a \wedge b) \vee d$.

For later we also record a related definition: a frame L is *normal* if the supremum $1 = x \vee y$ can be disjointified, as above (not necessarily with compact elements). We point out that if L is algebraic, then it has disjointification precisely when $\downarrow c$ is normal, for each compact c .

The following lemma is well known; see [Mo54] and [ST93, Lemma 2.1], where a proof is given.

Lemma 1.9. *Suppose that L is an algebraic frame with disjointification. Then, for any $p \in \text{Spec}(L)$, $\uparrow p$ is a chain. The converse is true if L has the FIP.*

Next, a lemma which culminates with the result we want.

Lemma 1.10. *Suppose that L is an algebraic frame. Then, for each disjoint pair $x, y \in L$, $c(x) \wedge c(y) = 0$. If L has disjointification, then*

- (a) $\text{CP}(L)$ also has disjointification;
- (b) $p \in L$ is prime if and only if $c \wedge d = 0$, with $c, d \in \mathfrak{k}(L)$, implies that $c \leq p$ or $d \leq p$.
- (c) $\text{Spec}(\text{CP}(L)) = \text{Spec}(L) \cap \text{CP}(L)$.

Proof. Note that $(\cdot)^{\perp\perp}$ is a nucleus, and so if $x \wedge y = 0$,

$$c(x) \wedge c(y) \leq x^{\perp\perp} \wedge y^{\perp\perp} = (x \wedge y)^{\perp\perp} = 0,$$

whence $c(x) \wedge c(y) = 0$, as asserted.

(a) Assume that a and b are compact in L , and suppose that $c, d \in \mathfrak{k}(L)$ are disjoint and witness the disjointification of a and b . By the first claim, $c(c)$ and $c(d)$ are disjoint, and $c(a) \vee c(b) = c(a) \vee c(d) = c(b) \vee c(c)$. Since every compact element of $\text{CP}(L)$ is of the form $c(g)$ for some compact $g \in L$, this suffices to establish that $\text{CP}(L)$ has disjointification.

(b) is straightforward and is left to the reader.

(c) The containment of $\text{Spec}(L) \cap \text{CP}(L)$ in $\text{Spec}(\text{CP}(L))$ is obvious. Now suppose that $q \in \text{Spec}(\text{CP}(L))$. According to (b), if a and b are compact in L and disjoint, then so are $c(a)$ and $c(b)$, which means that one of the two must be exceeded by q . But then either $a \leq q$ or $b \leq q$, proving that q is prime in L . ■

Theorem 1.11. *For any algebraic frame L with disjointification, c is an inductive nucleus.*

Proof. Since $\text{CP}(L)$ is algebraic, it suffices to show that, for each $x, y \in L$ and $p \in \text{CP}(L)$ which is prime (unambiguously, by Lemma 1.10), $c(x \wedge y) \leq p$ if and only if $c(x) \wedge c(y) \leq p$. As $c(x \wedge y) \leq c(x) \wedge c(y)$ is clear, the sufficiency is too. Conversely, if $c(x \wedge y) \leq p$, then $x \wedge y \leq p$, and since p is prime in L , either $x \leq p$ or $y \leq p$, and so $c(x) \leq c(p) = p$ or $c(y) \leq p$. Either way, $c(x) \wedge c(y) \leq p$. ■

Finally, in this section, we have the following remark.

Remark 1.12. Obviously, if L is completely distributive, then $\text{FP}(L) = \text{CP}(L)$. The reader is referred to [ST93], where these frames are characterized in a number of ways, under the assumption that L has disjointification. One of these conditions, equivalent to the complete distributivity of L , is the following: *For each compact $c \in L$, $\downarrow c$ has finitely many maximal elements.*

A slightly more general account of this may be found in [M72, Theorem 1.2].

One should also observe that, while the complete distributivity in L is certainly sufficient for $\text{FP}(L)$ and $\text{CP}(L)$ to coincide, it is far from necessary. The reader is referred to Example 5.7 below, in which there are no nonzero compact elements c such that $\downarrow c$ has finitely many maximal elements.

2 The Main Example.

As has already been stated in the introduction, the study of the constructs $\text{CP}(L)$ and $\text{FP}(L)$ was motivated by concerns about lattice-ordered groups (abbr. ℓ -groups). On the other hand, it seems excessive to digress into an introduction to the theory of ℓ -groups in this paper. We think the reader will find the presentation of the example for which $\text{FP}(L) \neq \text{CP}(L)$ complete enough to understand the essence of it, and interesting enough to persuade the reader to look up the references given below to fill in any gaps.

Definition & Remarks 2.1. It is well known that, for any ℓ -group G , the set $\mathcal{C}(G)$ of all the convex ℓ -subgroups of G is an algebraic frame under inclusion, and, indeed, that it is a complete sublattice of the lattice of all the subgroups of G . It is well known that $\mathcal{C}(G)$ has the FIP as well as disjointification. These facts, and other general references to the theory of ℓ -groups which we shall need, may be found in either [BKW77] or [D95].

In the context of ℓ -groups we use the notation $\mathcal{P}(G) = P(\mathcal{C}(G))$, $\mathcal{C}_f(G) = \text{FP}(\mathcal{C}(G))$, and $\mathcal{C}_c(G) = \text{CP}(\mathcal{C}(G))$.

Here is the example.

Example 2.2. *An intersection of a join of polars need not be of that form.*

An example is constructed of an abelian ℓ -group G in which there is a descending sequence of joins of polars, $(Q_n)_{n < \omega}$, whose intersection, $C = \bigcap_{n < \omega} Q_n$, is not in $\mathcal{C}_f(G)$. Thus, $\mathcal{C}_f(G)$ is not closed under countable meets of $\mathcal{C}(G)$, and it follows that $\mathcal{C}_f(G)$ is properly contained in $\mathcal{C}_c(G)$.

There is one construction from the theory of ℓ -groups that we will highlight here, prior to constructing the example. Let Λ be a *root system*; that is, a poset in which no two incomparable elements have a common lower bound. $V(\Lambda, \mathbb{R})$ is the subgroup of all real-valued functions f defined on Λ , for which

$$\text{coz}(f) \equiv \{ \lambda \in \Lambda : f(\lambda) \neq 0 \}$$

satisfies the ascending chain condition. It is well known that $V(\Lambda, \mathbb{R})$ is an ℓ -group, in which $f > 0$ if and only if $f(\lambda) > 0$, for each maximal $\lambda \in \text{coz}(f)$.

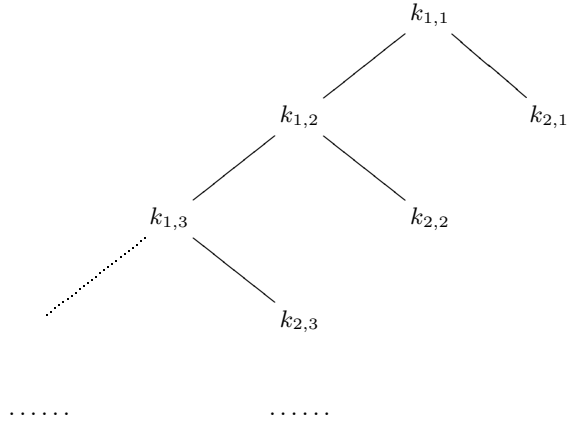
The presentation of the example is broken up into several steps:

1. We begin with C , the ℓ -group of all the convergent real sequences, ordered point-wise. This ℓ -group has a strong order unit ([D95, 7.4]), namely, the constant sequence 1; this means that the frame $\mathcal{L} \equiv \mathcal{C}(C)$ is compact. From a topological point of view, C is the ring $C(\alpha\mathbb{N})$ of all continuous functions on the compact space $\alpha\mathbb{N}$, which denotes the one-point compactification of the discrete set of natural numbers.
2. In \mathcal{L} each polar K is uniquely determined by a subset $S \subseteq \mathbb{N}$, according to the following correspondence:

$$K = K(S) \equiv \{f \in C : f|_S = 0\},$$

where $f|_S$ means that the function is restricted to S . Notice that the polar K of \mathcal{L} is complemented in \mathcal{L} if and only if S is finite or cofinite. In fact,

- (a) if S is infinite and has an infinite complement, then K consists entirely of sequences that converge to zero, and
 - (b) if S is cofinite, then each member of K has finite support.
3. Now let Λ be the following root system, pictured below. Λ is the disjoint union of two copies of \mathbb{N} , labelled N_1 and N_2 , with the elements enumerated by $N_i = \{k_{i,n} : n < \omega\}$, for $i = 1, 2$. Λ is ordered as follows:
 - (a) $k_{1,n} > k_{2,n}$, for each $n < \omega$;
 - (b) $m < n < \omega$ implies that $k_{1,m} > k_{1,n}$.



Now G is the ℓ -subgroup of $V \equiv V(\Lambda, \mathbb{R})$ consisting of all functions f in V such that $f|_{N_2}$ belongs to C . It is routine to verify that G is indeed an ℓ -subgroup of V . We identify C with the isomorphic copy consisting of the convex ℓ -subgroup of all $f \in G$ such that $f|_{N_1} = 0$.

4. Let χ_A denote the characteristic function of the subset $A \subseteq \Lambda$. Let

$$P_n = \chi_{\{k_{2,n}\}}^\perp = \{f \in G : f(m) = 0, \forall m \geq k_{2,n}\}.$$

Next, let

$$Q_j = \sum_{n \geq j}^\infty P_n = \{f \in G : f(k_{1,i}) = 0, \forall 1 \leq i \leq j\}.$$

The reader will have to verify that each P_n is minimal prime of $\mathcal{C}(G)$; this follows from the fact that the polar $\chi_{\{k_{2,n}\}}^{\perp\perp}$ is an atom in $\mathcal{P}(G)$, together with [M73, Lemma 2.1]. Each Q_j is then also prime, and therefore C is as well, as $C = \bigcap_{j < \omega} Q_j$, a meet of a chain of primes.

5. We claim that $C \notin \mathcal{C}_f(G)$. To the contrary, if C is a join of polars of G , then it must be a finite join of polars. So we suppose, by way of contradiction, that

$$C = K_1 + K_2 + \cdots + K_t,$$

where each $K_i \in \mathcal{P}(G)$.

Of course, each K_i is also a polar of C itself. By the remarks in 2 above, there exist subsets S_1, S_2, \dots, S_t of \mathbb{N} such that

$$K_i = K(S_i) = \{f \in C(\alpha\mathbb{N}) : f|_{S_i} = 0\}.$$

Note, also by 2, at least one of the S_i must be either finite or cofinite. In fact, if S is a finite subset of \mathbb{N} , then, regarding it as a subset of N_2 , we have that

$$\chi_S^\perp = \{f \in G : \exists j \in S, f(m) = 0, \forall m \geq j\},$$

and we point out that χ_S^\perp is not contained in C , but

$$\chi_S^\perp \cap C = K(S).$$

Conclusion: if S is a finite subset of \mathbb{N} , then $K(S)$ is **not** a polar of G .

We may therefore rearrange the summands in C as follows: $C = C_1 + C_2$, where $C_1 = K(S_1) + \cdots + K(S_l)$, with each S_j cofinite ($1 \leq j \leq l$), and $C_2 = K(S_{l+1}) + \cdots + K(S_t)$, with each S_j being infinite with an infinite complement.

Now let $u \in C$ be the function satisfying $u(k_{1,n}) = 0$, for all $n < \omega$, and $u(k_{2,n}) = 1$, for all $n < \omega$. Then $u = u_1 + u_2$, with $u_i \in C_i^\perp$, ($i = 1, 2$), and note that $u_1|_{N_2}$ has finite support, while $u_2|_{N_2}$ converges to zero.

This evident contradiction proves the claim.

6. As a final comment, observe the following: let

$$M = \{ f \in C : f|_{N_2} \text{ converges to } 0 \};$$

then $M \in \mathcal{C}_f(G)$; in fact, $M = K(O) + K(E)$, where O and E stand for the subsets of odd and even integers, respectively. Furthermore, $M = \bigwedge_{j < \omega} Q_j$, with the infimum induced in $\mathcal{C}_f(G)$. Thus, while M is (completely) meet irreducible in $\mathcal{C}(G)$, it is a meet of strictly larger primes in $\mathcal{C}_f(G)$.

3 Branched Primes.

In this section we examine the relationship between joins of polars and meets of maximal elements. Throughout this section L stands for an algebraic frame with the FIP. Eventually it will be assumed that L is coherent.

We begin by introducing a construction which is rather prominent in the study of rings of continuous functions ([GJ76, 4I]). Consider a *Tychonoff* space X ; that is to say, a Hausdorff space in which the cozerosets of the space form a base for the frame of open sets $\mathfrak{D}(X)$. Let $C(X)$ denote the ring of all continuous real-valued functions defined on X , with pointwise operations of addition, multiplication, supremum and infimum. Regarded in this way, $C(X)$ is a commutative ring with identity – the constant function 1 – as well as a lattice-ordered group.

Consider now an ideal I of $C(X)$ having the property that if $f \in I$ and f vanishes at $p \in X$, then it vanishes on a neighborhood of p . Such ideals are called *pure* in the literature. They are intersections of minimal prime ideals. These ideals are thoroughly examined in [DM83].

This marks the point of transition into the work of [MZ06b], as explained in the introduction, for the pure ideals are none other than the regular elements in an appropriately chosen frame of ideals. In this article, the construct we are about to define is used for a different purpose.

Definition 3.1. For $p \in \text{Spec}(L)$, put

$$O(p) = \vee \{ a^\perp : a \in \mathfrak{k}(L), a \not\leq p \}.$$

We have the following elementary result. We leave the proof of (a) and (b) to the reader.

Proposition 3.2. Let $p, q \in \text{Spec}(L)$;

- (a) if $p \leq q$, then $O(q) \leq O(p)$.
- (b) $O(p)$ is a *d*-element.

- (c) $q \in \text{Min}(L)$ and $q \leq p$ imply that $O(p) \leq q$.
- (d) If $O(p) \leq q$ and q is minimal over $O(p)$ then $q \leq p$.
- (e) $O(p) = \wedge \{q \in \text{Min}(L) : q \leq p\}$.

Proof. (c) Assume that $q \in \text{Min}(L)$ and $q \leq p$. Note that in the definition of $O(p)$ the supremum in question is over an upward directed set. Thus, if $O(p) \not\leq q$, there exist disjoint compact elements a and b such that $a \not\leq p$ and $b \not\leq q$. But then $a \not\leq q$ as well, which is absurd.

(d) Suppose that $O(p) \leq q$ and q is minimal over $O(p)$. Suppose also that there is a compact element $c \leq q$ such that $c \not\leq p$. For each pair of compact $a \not\leq q$ and $b \not\leq p$, one easily checks that $a \wedge b > 0$. This implies that such a and b generate a filter F of compact elements, which is contained in an ultrafilter U . Putting

$$m = \vee \{a^\perp : a \in U\},$$

yields a minimal prime m (by [MZ03, Lemma 2.2]), which satisfies $O(p) \leq m \leq q$. By our assumption, it follows that $m = q$ and hence $c \leq m$, which is a contradiction, since $c \in F \subseteq U$. We therefore conclude that $q \leq p$, as asserted.

(e) easily follows from (c) and (d). ■

Next, we consider when a prime p lies in $\text{FP}(L)$. We say that a prime p is *branched* if it is not minimal, yet it is the supremum of the minimal primes beneath it.

Proposition 3.3. *Let $p \in \text{Spec}(L)$. Then*

$$p = \vee \{a_\lambda^\perp : \lambda \in \Lambda\}$$

for suitable compact $a_\lambda \in L$ if and only if it is minimal or branched.

Proof. Since every minimal prime is a supremum of polars of the form a^\perp , with a compact, it clear that if p is a branched prime, p must also be such a supremum.

Next, let $\{a_\lambda : \lambda \in \Lambda\}$ be a subset of compact elements, and suppose that $p = \vee \{a_\lambda^\perp : \lambda \in \Lambda\}$. Since $a_\lambda^\perp \leq p$, for each $\lambda \in \Lambda$, it follows that $a_\lambda \not\leq O(p)$. In view of Proposition 3.2, there is a minimal prime $q_\lambda \leq p$, such that $a_\lambda \not\leq q_\lambda$, whence $a_\lambda^\perp \leq q_\lambda$. Taking suprema over Λ , one easily sees that $p = \vee_{\lambda \in \Lambda} q_\lambda$, as promised. ■

We have the following immediate corollary. We do not know whether the conclusion can be strengthened to $L = \text{FP}(L)$.

Corollary 3.4. *Suppose that each prime of L is either minimal or branched. Then $L = \text{CP}(L)$.*

Definition & Remarks 3.5. Recall that L is *archimedean* if, for each $c \in \mathfrak{k}(L)$, $\bigwedge \text{Max}(\downarrow c) = 0$. This concept first appeared in [M73]; for an ampler discussion of it and the rest of this commentary, the reader is referred to [MZ03, §6]. Say that $x \in L$ is *upper-archimedean* if $\uparrow x$ is archimedean. Denote the set of all upper-archimedean elements of L by $\mathbf{a}^\uparrow(L)$. By [MZ03, Lemma 6.2], $\mathbf{a}^\uparrow(L)$ is closed under arbitrary meets, and so there is a nucleus ar on L such that $arL = \mathbf{a}^\uparrow(L)$.

Observe that if L is compact then $x \in \mathbf{a}^\uparrow(L)$ precisely when x is an infimum of maximal elements of L .

Finally, define $z \equiv \widehat{ar}$; that is, $z(x) = \vee \{ ar(c) : c \leq x, c \in \mathfrak{k}(L) \}$, for each $x \in L$. Observe that $zL = \text{Ab}(ar)$. The elements of zL are called *z-elements*.

We are now able to state another easy corollary of Proposition 3.3. As with the preceding corollary, we do not know whether $\text{FP}(L)$ can be substituted for $\text{CP}(L)$. We shall return to this issue in §5; however, even in the context of $C(X)$ we are unable to decide whether $zL \subseteq \text{FP}(L)$.

Corollary 3.6. *Suppose that L is coherent. Assume that each maximal element is either minimal or branched. Then $\mathbf{a}^\uparrow(L) \subseteq \text{CP}(L)$, whence $zL \subseteq \text{CP}(L)$.*

Remark 3.7. Proposition 3.3 gives rise to yet another construct, which is closely related to $\text{FP}(L)$. As we shall see in Example 5.8, however, it need not coincide with $\text{FP}(L)$.

Denote by $F_{\mathfrak{k}}(L)$ the set of all joins of polars of the form $\vee \{ a_\lambda^\perp : \lambda \in \Lambda \}$, with each a_λ compact. Since $a^\perp \wedge b^\perp = (a \vee b)^\perp$, for any two compact elements of L , it is easy to see that $F_{\mathfrak{k}}(L)$ is a subframe of $\text{FP}(L)$. Moreover, the reader should have no trouble verifying that $dL \subseteq F_{\mathfrak{k}}(L)$ precisely when $P(L) \subseteq F_{\mathfrak{k}}(L)$, which is so if and only if $F_{\mathfrak{k}}(L) = \text{FP}(L)$.

4 Distinctions in (\dagger) .

In this section we examine the string of inclusions

$$(\dagger) \quad P(L) \subseteq \text{Min}^*(L) \subseteq dL \subseteq \text{FP}(L) \subseteq \text{CP}(L),$$

first displayed in Proposition 1.6(b). We begin with the general context of the first section: L stands for an algebraic frame with the FIP. Recall that $\text{Min}^*(L) = dL$ if and only if dL is regular, which, by [MZ03, Proposition 5.2], is the case precisely when every prime d -element is minimal. Recall also – Proposition 1.6(c) – that if $dL = \text{FP}(L)$, then also $\text{FP}(L) = \text{CP}(L)$. Thus, for our main example in §2, we have $dL \subset \text{FP}(L) \subset \text{CP}(L)$.

Next, we observe when $P(L) = \text{Min}^*(L)$. We borrow notation from [M73]: $0 < b \in L$ is *basic* provided $\downarrow b$ is a chain. Then we refer the reader to [M73, Lemma 2.1], which

tells us that a polar $x \in L$ is prime if and only if it is a minimal prime, and that this happens precisely when $x = b^\perp$, for a suitable basic element b . Then the following is easy:

Proposition 4.1. *$P(L) = \text{Min}^*(L)$ if and only if $\text{Min}(L)$ is discrete in the hull-kernel topology.*

Proof. To begin, the base of open sets in the hull-kernel topology of $\text{Min}(L)$ may be described as the family of

$$c_{\text{Min}}(a) \equiv \{p \in \text{Min}(L) : a \not\leq p\},$$

for all compact a in L , and $c_{\text{Min}}(a)$ coincides with $\{p \in \text{Min}(L) : a^\perp \leq p\}$. Thus, $c_{\text{Min}}(a)$ is a singleton set if and only if a is basic.

Assume now that $P(L) = \text{Min}^*(L)$; then every minimal prime is a polar, and therefore of the form b^\perp , for some basic b . Moreover, it is routine to verify that b may be chosen compact. The comments in the first paragraph then imply that $\text{Min}(L)$ is discrete.

Conversely, if $\text{Min}(L)$ is discrete then every minimal prime is a polar (of a basic element), and it easily follows that every meet of minimal primes is also a polar. ■

Remark 4.2. It is worth remarking that an immediate consequence of Proposition 4.1 is the fact that $P(L)$ is an atomic boolean algebra, and, indeed, isomorphic to the power set of $\text{Min}(L)$.

Before proceeding with our exploration of distinctions in (\dagger) , it is important to state a general principle, which greatly simplifies matters in the application we have in mind in this section, but especially again in [MZ06b].

Let j be an inductive nucleus on L (1.1.9). Recall ([M06]) that a compact element a is said to be a j -unit if $j(a) = 1$. We then have the following straightforward result.

Proposition 4.3. *Suppose that j is an inductive nucleus such that every polar is a j -element, and jL is a sublattice of L . Then jL is a complete sublattice, and $T(L) = T(jL)$, for each of the operators T in (\dagger) . If there is a j -unit then all of the sublattices in (\dagger) may be computed in a coherent frame.*

Proof. Most of the claims of the proposition are clear enough, after one observes that, for each $x \in L$, $x \leq j(x) \leq x^{\perp\perp}$, which implies that $x^\perp = j(x)^\perp$, and that $P(L) = P(jL)$. This, together with the assumption that jL is a sublattice of L , implies the coincidence of $T(L)$ and $T(jL)$, for each of the other operators. ■

We have two applications of Proposition 4.3 in mind. The first concerns f -rings; the second is more specific to rings of continuous functions. Let us proceed with the setup. The term “ f -ring”, throughout, shall signify “commutative f -ring with 1”. For background on f -rings we refer the reader to [BKW77].

Remark 4.4. Recall that a lattice-ordered ring is an *f-ring* if for each disjoint pair $f \wedge g = 0$ and each $h \geq 0$ in A , we have $fh \wedge g = 0$. Armed with Zorn's Lemma, then A is an *f-ring* if and only if it is a subdirect product of totally ordered rings ([BKW77, Theorem 9.1.2]).

In an *f-ring* A , consider the nucleus ℓ , which computes, given $C \in \mathcal{C}(A)$ the ℓ -ideal $\ell(C)$ generated by C . (To explain the term *ℓ-ideal*: it denotes a ring ideal that is also a convex ℓ -subgroup.) It is well known and easy to check that

$$\ell(C) = \{ r \in A : |r| \leq fa, \text{ for some } 0 \leq f \in A, 0 \leq a \in C \}.$$

ℓ is an inductive nucleus and the frame $\ell\mathcal{C}(A) \equiv \mathcal{C}_\ell(A)$ of ℓ -ideals is a sublattice of $\mathcal{C}(A)$, as the sum of ℓ -ideals is always an ℓ -ideal. Moreover, every polar is an ℓ -ideal, by the definition of “*f-ring*”.

The import of Proposition 4.3 here is that, if we are interested in questions regarding $\text{FP}(\mathcal{C}(A))$ and $\text{CP}(\mathcal{C}(A))$, then we may as well carry out calculations in $\mathcal{C}_\ell(A)$. Note that the identity 1 of A generates A as an ℓ -ideal; that is, $\mathcal{C}_\ell(A)$ is coherent.

Let X be a Tychonoff space. The ring $C(X)$ is an archimedean *f-ring*. In consideration of the frame $\mathcal{C}(C(X))$ of convex ℓ -subgroups, as in 4.4, one may make a further reduction (apart from working with ℓ -ideals).

To proceed with this reduction we make an additional assumption about the *f-ring* A : that it is an archimedean *f-algebra* – and by the latter term we understand that A is both an *f-ring* and a real vector lattice, and that for all $r \in \mathbb{R}$, and $f, g \in A$, we have $r(fg) = (rf)g$. (The reader is referred to [HuP82] for further background concerning these remarks.)

Remark 4.5. Starting with $\mathcal{C}(A)$, we pass to the frame of upper-archimedean elements $\mathbf{a}^\uparrow(\mathcal{C}(A))$. By [HuP80a, Theorem 2.1], $\mathbf{a}^\uparrow(\mathcal{C}(A))$ consists of the uniformly closed convex ℓ -subgroups of A . They are ℓ -ideals ([HuP82, Proposition 2.1(ii)]). It is also well known that polars in an archimedean frame are upper-archimedean; see [MZ03, §6]. Next, observe that the z -elements of $\mathcal{C}(A)$ are the z -ideals of [HuP80a, HuP80b, Pa81]; we denote, for consistency, $z\mathcal{C}(A) \equiv \mathcal{C}_z(A)$. The preceding comments signify that $\mathcal{C}_z(A) \subseteq \mathcal{C}_\ell(A)$, and that every polar of A is a z -ideal. Proposition 4.3 again tells us that computations regarding the items in (†), for convex ℓ -subgroups of an archimedean *f-ring* A , may be made in $L = \mathcal{C}_z(A)$, as long as in A the sum of two z -ideals is a z -ideal. As to that, note that in any uniformly complete *f-algebra*, $\mathcal{C}_z(A)$ is a sublattice of $\mathcal{C}(A)$ ([HuP80a, Theorem 3.5]). For any Tychonoff space X , one may simply note that $C(X)$ is a uniformly complete *f-algebra*, or, alternatively, obtain that the sum of any two z -ideals in $C(X)$ is a z -ideal concretely from [GJ76, Lemma 14.8].

As with $\mathcal{C}_\ell(A)$, $\mathcal{C}_z(A)$ is a coherent frame. Note that both $\mathcal{C}_\ell(A)$ and $\mathcal{C}_z(A)$ have disjointification and are therefore normal frames.

Thus, abstractly, if it is the convex ℓ -subgroups of a uniformly complete *f-algebra* we are interested in, then, by Proposition 4.3, our considerations regarding $\text{FP}(L)$ and

$CP(L)$ may be carried out in a coherent normal, Yosida frame (see [MZ06a]; an algebraic frame L is *Yosida* if every compact element of L is a meet of maximal elements).

5 Applications in $C(X)$.

Throughout this section X designates a Tychonoff space. In the context of the f -ring $C(X)$, with the remarks of 4.5 in mind, (\dagger) becomes

$$(\dagger_C) \quad \mathcal{C}_d(X) \subseteq \mathcal{C}_f(X) \subseteq \mathcal{C}_c(X) \subseteq \mathcal{C}_z(X),$$

where $\mathcal{C}_d(X) \equiv d\mathcal{C}(C(X))$, $\mathcal{C}_f(X) \equiv \mathcal{C}_f(C(X))$, $\mathcal{C}_c(X) \equiv \mathcal{C}_c(C(X))$, and $\mathcal{C}_z(X) \equiv \mathcal{C}_z(C(X))$. Here we examine coincidences in (\dagger_C) , beginning with the version of Corollary 3.6 given in Proposition 5.2. First, we must take a closer look at $\mathcal{C}_z(X)$.

Remark 5.1. As outlined in [MZ05, 4.1], the compact elements in $\mathcal{C}_z(X)$ are of the form

$$\langle f \rangle_z = \{ g \in C(X) : \text{coz}(g) \subseteq \text{coz}(f) \} \quad (f \in C(X)).$$

As usual, $\text{coz}(f) = \{ x \in X : f(x) \neq 0 \}$, i.e., the *cozeroset* of f . The *zeroset* of f , denoted $Z(f)$, is the complement of $\text{coz}(f)$. Then observe that

$$\langle f \rangle_z = \bigcap_{x \in Z(f)} M_x,$$

where M_x stands for the maximal ideal $M_x = \{ f \in C(X) : f(x) = 0 \}$.

The above expression for each $\langle f \rangle_z$ witnesses the fact that $\mathcal{C}_z(X)$ is a Yosida frame. The same expression also gives us the following corollary of Proposition 3.3.

We say that a point x in the space X is *branched* if M_x is a branched prime. Recall that $x \in X$ is a *P-point* if and every continuous function which vanishes at x also vanishes on a neighborhood of x . Note then that M_x is a minimal prime if and only if x is a *P-point*; (see [GJ76, 4L & Theorem 14.29].)

Proposition 5.2. *Assume that each point of X is either a P-point or branched. Then $\mathcal{C}_c(X) = \mathcal{C}_z(X)$.*

Remark 5.3. When should we expect the hypothesis of Proposition 5.2 to be satisfied? Le Donne (in [Ld77, Proposition 2]) shows that if X is first countable then every nonisolated point is branched. (The reader should note that in a first countable space every *P-point* is isolated.) According to the main theorem in [Ld77, §3], in a metric space every nonminimal prime z -ideal is branched.

Recall that X is an *almost P -space* if every nonempty zeroset of X has nonempty interior.

Proposition 5.4. $\mathcal{C}_d(X) = \mathcal{C}_z(X)$ if and only if X is an almost P -space. Thus, for these spaces,

$$\mathcal{C}_d(X) = \mathcal{C}_f(X) = \mathcal{C}_c(X) = \mathcal{C}_z(X).$$

Proof. Combine [HuP80a, Theorem 5.4(ii)] with [HuP80b, Theorem 10.3(ii)]. ■

Next, we ask when

$$(\ddagger) \quad \mathcal{C}_d(X) = \mathcal{C}_f(X) = \mathcal{C}_c(X).$$

The reader should keep in mind that, for any algebraic frame L , $dL = \text{FP}(L)$ implies that $dL = \text{CP}(L)$ (Proposition 1.6(c)).

To answer this question we need to recall the concept of o -convergence in the context of archimedean vector lattices.

Definition 5.5. Let G be an archimedean vector lattice. Recall that the sequence $(g_n)_{n < \omega}$ in G *o -converges* to $g \in G$ if there is a decreasing sequence u_n with $\bigwedge_n u_n = 0$ such that $|g - g_n| \leq u_n$. The sequence $(g_n)_{n < \omega}$ is *o -Cauchy* if there is a decreasing sequence u_n with $\bigwedge_n u_n = 0$ such that, for every $k \in \mathbb{N}$, $|g_{n+k} - g_n| \leq u_n$. Finally, G is *order complete* if every o -Cauchy sequence of G o -converges.

The reader should also review [DHH80] for a representational approach to the proof of the following proposition.

Proposition 5.6. Let X be a Tychonoff space. Then the identities in (\ddagger) hold if and only if X is a quasi F -space.

Proof. In [Pa81, Theorem 11.2] it is shown that the uniformly complete vector lattice G is order complete precisely when the sum of two d -subgroups is a d -subgroup; that is, when $\mathcal{C}_d(G) = \mathcal{C}_f(G)$.

Next, recall that X is a quasi F -space when every dense cozeroset is C^* -embedded. It is shown by combining [Pa81, Theorem 11.8 & Lemma 11.9] that a Tychonoff space X is quasi F if and only if $C(X)$ is order complete. ■

We conclude the section with two examples.

Example 5.7. Tychonoff spaces for which $\mathcal{C}_c(X) \neq \mathcal{C}_z(X)$.

Applying Propositions 1.6(d) and [MZ06b, 8.4], we see that if an algebraic frame with the FIP has the property that

$$L = a^{\perp\perp} \vee a^\perp, \quad \text{for each compact } a \in L,$$

then $\text{Min}^*(L) = dL = \text{FP}(L) = \text{CP}(L)$; borrowing from the theory of ℓ -groups, we call frames with this feature *projectable*. For $L = \mathcal{C}(C(X))$, it is well known that L is projectable if and only if X is *basically disconnected*; that is, the closure of every cozeroset of X is clopen. Now pick X to be any basically disconnected space which is not an almost P -space – for example, $\beta\mathbb{N}$. Then

$$\text{Min}^*(L) = \mathcal{C}_d(X) = \mathcal{C}_f(X) = \mathcal{C}_c(X) \neq \mathcal{C}_z(X).$$

Example 5.8. *An example in which $\text{FP}(L) \neq F_{\mathfrak{k}}(L)$.*

Refer to the discussion in 3.7. It is well known – see, for example [GJ76, Theorem 14.27] – that the Stone-Ćech remainder $X = \beta\mathbb{N} \setminus \mathbb{N}$ has the feature that in $L = \mathcal{C}(C(X))$ every prime exceeds a unique minimal prime. Yet it is also well known that X is an almost P -space. Thus, by Proposition 5.4, $dL = \text{FP}(L) = \text{CP}(L) = zL$.

On the other hand, there are no branched primes, yet L has maximal elements m which are not minimal. This means that $F_{\mathfrak{k}}(L) \neq \text{FP}(L)$, as each such $m \in \text{FP}(L) \setminus F_{\mathfrak{k}}(L)$.

6 Open Questions.

Around the issue of distinguishing between $\text{FP}(L)$ and $\text{CP}(L)$, several interesting open questions remain.

Question 6.1. *If L is an archimedean frame with the FIP, is $\text{FP}(L) = \text{CP}(L)$?*

The question is open even for $L = \mathcal{C}(G)$, the frame of convex ℓ -subgroups of an archimedean ℓ -group. And recall that if G is also an f -ring, then by the remarks in 4.4, we may without loss of generality assume $L = \mathcal{C}_{\ell}(G)$.

This question is open for rings of continuous functions, and in this context one asks: *Is $\mathcal{C}_f(X) = \mathcal{C}_c(X)$, for every Tychonoff space?*

Question 6.2. *Is $\text{FP}(L)$ always an algebraic lattice? When $\text{FP}(L)$ is an algebraic lattice, is $\text{FP}(L) = \text{CP}(L)$?*

Question 6.3. *Characterize the Tychonoff spaces for which $\mathcal{C}_c(X) = \mathcal{C}_z(X)$. Does it always follow from this condition that $\mathcal{C}_f(X) = \mathcal{C}_z(X)$?*

Note that $\mathcal{C}_f(X) = \mathcal{C}_z(X)$ if and only if every uniformly closed convex ℓ -subgroup of $C(X)$ is a join of polars.

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