New Torsion Theory in Unital Abelian ℓ-Groups

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For Charles Holland, on his 70th or his 75th birthday,
whichever he prefers!

ABSTRACT. This paper introduces the notion of a *functorial torsion class (FTC)*: in a concrete category $\mathfrak{C}$ which has image factorization, one considers monocoreflective subcategories which are closed under formation of subobjects.

Here the interest is in FTCs in the category of abelian lattice-ordered groups with designated strong order unit. The FTCs $\mathfrak{T}$ consisting of archimedean lattice-ordered groups are characterized: for each subgroup A of the rationals with the identity 1, either $\mathfrak{T} = \mathfrak{S}(A)$, the class of all lattice-ordered groups of functions on a set X which have finite range in A, or $\mathfrak{T} = \mathbb{T}(A)$, the class of all subgroups of A with 1.

As for FTCs possessing non-archimedean groups, it is shown that if $\mathfrak{T}$ is an FTC containing a subgroup A of the reals with 1, of rank two or greater, then $\mathfrak{T}$ contains all ℓ-groups of the form $A \overrightarrow{\times} G$, for all abelian lattice-ordered groups G. Finally, the least FTC that contains a non-archimedean group is the class of all $\mathbb{Z} \overrightarrow{\times} G$, for all abelian lattice-ordered groups G.

1 Introduction.

At a workshop in 2008, Charles Holland asked the author how the concept of a torsion class of lattice-ordered groups – abbr. ℓ-groups, as is customary – might be adapted to the context of ℓ-groups with strong order unit, or to MV-algebras. This article is, in a sense, an answer to Holland's question, although not the answer he might have expected. These are not the torsion classes of ℓ-groups, developed since the seventies, since [M75].

The deeply rooted link between ℓ-groups and frames notwithstanding, this paper is about ℓ-groups, and about abelian ℓ-groups with a designated strong unit, in particular.

For background on lattice-ordered groups we refer the reader to [BKW77, D95]. What is recorded in the following is there for the reader's convenience. All ℓ-groups in this article are assumed to be abelian.

Definition & Remarks 1.1. For the record, $(G, +, 0, -(\cdot), \vee, \wedge)$ is a *lattice-ordered group* (abbreviated *ℓ-group*) if $(G, +, 0, -(\cdot))$ is a group with $(G, \vee, \wedge)$ as an underlying lattice, and the following distributive law holds:

$$a + (b \vee c) = (a + b) \vee (a + c).$$

The above then implies the corresponding distributive law for sum over infimum. The elements of G for which $g \geq 0$ are said to be *positive*; the set of positive elements of G is denoted G^+.

We recite the information to be used in this article; in the sequel G stands for an ℓ-group.

1. The underlying lattice of an ℓ-group is distributive ([D95, Corollary 3.17]), and the group structure is torsion free ([D95, Propositions 3.15 & 3.16]).

2. A subgroup of G is called an *ℓ-subgroup* if it a sublattice as well. The ℓ-subgroup C is *convex* if $a \leq g \leq b$ with $a, b \in C$ implies that $g \in C$. Let $\mathcal{C}(G)$ denote the lattice of all convex ℓ-subgroups of G. $\mathcal{C}(G)$ is a complete sublattice of the lattice of all subgroups of G ([D95, Theorem 7.5]), and it is distributive; the latter is due to G. Birkhoff ([D95, Proposition 7.10]).

   In $\mathcal{C}(G)$ the convex ℓ-subgroup generated by $a \in G$ is denoted $\langle a \rangle_G$. Each compact element of $\mathcal{C}(G)$ is of this form; this is a restatement of [D95, Proposition 7.16]. Note that, for $0 \leq a, b \in G$, $\langle a \rangle_G \subseteq \langle b \rangle_G$ precisely when $a \leq nb$, for a suitable natural number n.

2 Torsion Theory.

Classical torsion theory, as in [L71], motivates the definition that follows. There is, however, no mention of injectives; as the reader may already be aware, categories of ℓ-groups are typically poor in injectives.

Definition 2.1. We assume the context of a concrete category $\mathfrak{C}$ in which the morphisms have image factorization. The terms "class" and "full subcategory" will be used interchangeably throughout this article.

A class of $\mathfrak{C}$-objects $\mathfrak{T}$ is called a *functorial torsion class* if it is closed under formation of coproducts, subobjects and images under surjective maps of $\mathfrak{C}$. We will use the abbreviation *FTC* throughout for "functorial torsion class".

The reader who is well versed in category theory will recognize in closure under taking copoducts and images under surjective maps that FTCs are monocoreflective. Without getting technical about the matter, we observe here that a monocoreflection occurs relative to paired classes of monomorphisms and epimorphisms of $\mathfrak{C}$, with respect to which the parent category has unique factorization. In the context of this article the monics will be the one-to-one maps of the category, and the epics the surjective maps. The converse is also true: if $\mathfrak{T}$ monocoreflective in $\mathfrak{C}$, then it is closed under the formation of coproducts and images under surjective maps of $\mathfrak{C}$.

Let us formally define coreflection. We refer the reader to [HS79] for any unexplained material.

Definition & Remarks 2.2. Let $\mathfrak{A}$ denote a category. We say that a full subcategory $\mathfrak{C}$ is *coreflective* if for each $\mathfrak{A}$-object A there is a $\mathfrak{C}$-object τA and a morphism $\tau_A : \tau A \longrightarrow A$, such that for each $\mathfrak{C}$-object X and each map $h : X \longrightarrow A$ there is a unique map $\hat{h} : X \longrightarrow \tau A$ auch that $\tau_A \cdot \hat{h} = h$.

It is well known that τ defines a covariant functor from $\mathfrak{A}$ into $\mathfrak{C}$ and that it is the left adjoint of the inclusion functor. τ is the *coreflection*, and if each τ_A is a monomorphism, τ is called a *monocoreflection.* The subcategory $\mathfrak{C}$ is said to be *monocoreflective* in $\mathfrak{A}$.

If $\mathfrak{C}$ is coreflective in $\mathfrak{A}$, then $\mathfrak{C}$ preserves the colimits of $\mathfrak{A}$ ([HS79, 28.3]).

Although we shall save any remarks about how the results of this paper translate to MV-algebras, we will (conveniently) observe here that because the category of abelian ℓ-groups with designated strong unit is equivalent to that of MV-algebras, and the latter is a variety, then both are complete and cocomplete.

Let us conclude this section by pointing out that what we have elected to call an FTC is (categorically) a *covariety.*

3 FTCs of Archimedean ℓ-Groups: Reductions.

For the rest of the paper, the ambient category will be $\ell\,\mathfrak{Ab}^*$, the category of all abelian ℓ-groups with a positive designated strong order unit, which will be denoted by 1, with all ℓ-homomorphisms that preserve the designated unit.

An ℓ-group with a designated unit is a *unital* ℓ-group. We will also use the terms *unital ℓ-subgroup, unital quotient,* and *unital coproduct* to indicate that, respectively, an ℓ-subgroup inherits the designated unit, a surjective ℓ-homomorphism preserves the designated unit, and that the coproduct in question is taken in $\ell\,\mathfrak{Ab}^*$.

We now consider coproducts in $\ell\,\mathfrak{Ab}^*$. The reader is cautioned from the start that this is different from the coproduct in the category of all abelian ℓ-groups, which is discussed in [M73a] and [BCPT90]. The spirit of the results is in keeping with those in the latter reference, however.

"Free product" and "coproduct" are employed synonymously here. This coproduct is the construct of [Mu88], although our arguments are different. The coproduct of the empty collection exists, in any category, if and only if there is an initial object ([HS79, 20.3]). In $\ell\,\mathfrak{Ab}^*$ the initial object is $\mathbb{Z}$, and so every monocoreflective subcategory of

$\ell\,\mathfrak{Ab}^*$ contains $\mathbb{Z}$. In particular, the least FTC is the one comprising $\mathbb{Z}$ and $\{0\}$. We refer to this as the *trivial* FTC.

$\ell\,\mathfrak{Ab}^*(A,B)$ denotes the set of morphisms in $\ell\,\mathfrak{Ab}^*$ between A and B. Recall that a group which can be embedded in $\mathbb{Q}$, the additive group of rationals, is said to have *rank 1*. Note that if A has rank 1, then $|\ell\,\mathfrak{Ab}^*(A,G)| \leq 1$, for each $\ell\,\mathfrak{Ab}^*$-object G, and we have equality if G is divisible.

Here is the central "lemma", which seems interesting in its own right. The blanket assumption of divisibility is there to smooth out the proofs. The reader ought to perhaps review two standard theorems from the theory of abelian ℓ-groups:

1. **The Conrad-Harvey-Holland Embedding Theorem.** Every abelian ℓ-group may be embedded in one of the form $V(\Lambda,\mathbb{R})$, where Λ denotes a root system – that is, a poset in which no two incomparable elements have a common lower bound – and $V = V(\Lambda,\mathbb{R})$ stands for the set of all real-valued functions f on Λ for which the cozeroset

   $$\mathrm{coz}(f) = \{\,\lambda\,:\,f(\lambda) \neq 0\,\}$$

   satisfies the ascending chain condition. V is a subgroup of the group of all real-valued functions on Λ, and an ℓ-group under the ordering set by $f \geq 0$ provided f is zero, or $f(\lambda) > 0$, for each maximal $\lambda \in \mathrm{coz}(f)$ (see [D95, Theorems 51.3 and 51.7]) .

2. **The Yosida Representation.** (For unital ℓ-groups; we refer the reader to [BKW77, Corollary 13.2.6]): Let G be a unital ℓ-group. The *Yosida space* Y is the space of all maximal convex ℓ-subgroups, endowed with the hull-kernel topology. It is well known that Y is a compact Hausdorff space.

   Consider the assignment, for $f \in G$, $f \mapsto \sigma(f)$ where $\sigma(f)$ denotes the continuous real-valued function

   $$\sigma(f)(\mathfrak{m}) = \mathfrak{m} + f,\ \ \mathfrak{m} \in Y$$

   and we may assume without loss of generality that $\mathfrak{m} + 1 = 1$, where $1 \in G$ is the designated unit of G. This representation embeds G in $C(Y)$ if and only if G is archimedean.

Theorem 3.1. *Let A be an $\ell\,\mathfrak{Ab}^*$-object. The following are equivalent.*

(a) $|\ell\,\mathfrak{Ab}^*(A,G)| = 1$*, for every divisible $\ell\,\mathfrak{Ab}^*$-object G.*

(b) *A has rank 1.*

(c) *For each divisible $\ell\,\mathfrak{Ab}^*$-object G, $A \coprod G = G$.*

Moreover, if A is a unital subgroup of $\mathbb{R}$ which is not rank 1, then $A \coprod A$ is not hyperarchimedean.

Proof. We have already noted that (b) implies (a).

Assume (b) and that G is divisible. Observe that the divisible hull of A is $\mathbb{Q}$, which, in turn, has only one possible $\ell\mathfrak{Ab}^*$-embedding in G. Therefore, we have a canonical embedding $u_A : A \longrightarrow G$; let $u_G : G \longrightarrow G$ be the identity. If $v_A : A \longrightarrow H$ and $v_G : G \longrightarrow H$ are $\ell\mathfrak{Ab}^*$-morphisms, then, by the above comments, v_A is the restriction of v_G to A. Thus, it is clear that $G = A \coprod G$, and (b) implies (c).

If (c) holds, then $A \coprod dA = dA$, and f and g are morphisms of $\ell\mathfrak{Ab}^*$ defined on dA, then $f_A = g_A$ (by the universality of the coproduct), and so $f = g$. Therefore, dA and, in turn A, satisfies (a).

We shall prove that (a) implies (b) through three reductions.

1. We use the Conrad-Harvey-Holland Theorem: A is $\ell\mathfrak{Ab}^*$-embedded in $V = V(\Lambda, \mathbb{R})$, for some root system Λ in which each member is exceeded by a maximal element. There are three cases to consider. Let e be the unit of A; we may assume that $e(\lambda) = 1$ at every maximal λ.

   (a) *$e(\mu) = 0$ at each nonmaximal μ.* Fix a nonmaximal $\nu \in \Lambda$ and let h be the $\ell\mathfrak{Ab}^*$-automorphism of V which doubles $f(\nu)$ of each $f \in V$. Then h and the identity map differ on A.

   (b) *For some nonmaximal μ, $e(\mu) > 0$:* Without loss of generality, one may take $e(\mu) = 1$. Let h be the $\ell\mathfrak{Ab}^*$-automorphism of V which sends $f(\mu)$ to $2f(\mu) - f(\alpha)$, where α is the unique maximal element above μ. It is straightforward to verify that h is an $\ell\mathfrak{Ab}^*$-automorphism which differs from the identity on A.

   (c) *For some nonmaximal μ, $e(\mu) < 0$:* The argument mirrors the preceding one.

   The conclusion must be that Λ is trivially ordered, and A is archimedean.

2. We may assume that A is $\ell\mathfrak{Ab}^*$-embedded in $C(Y)$, with Y being the Yosida space of A, and note that the embedding separates the points of Y. This implies that for distinct points x and y, the two evaluation maps e_x and e_y are distinct on A. This contradicts (b), unless $|Y| = 1$, and A is a subgroup of $\mathbb{R}$.

3. Suppose, by way of contradiction, that A contains an irrational number, and for the moment assume that A is divisible. Then $A = Q \oplus B$, where Q is the rational subspace of A generated by 1, and observe that $B \neq \{0\}$.

   Let $G = A \times \mathbb{R}$, lexicographically ordered from the left. Define two maps $t_1, t_2 : A \longrightarrow G$ by $t_1(q, b) = (q, b, b)$ and $t_2(q, b) = (q, b, -b)$. It is easily checked that t_1 and t_2 are distinct $\ell\mathfrak{Ab}^*$-morphisms, again producing a contradiction.

4. Thus, with the hypothesis of (a) and A divisible we are forced to conclude that $A \cong \mathbb{Q}$. In general, if A satisfies (a), then so does its divisible hull, which means that A has rank 1.

The reader should also realize that, with A a subgroup of $\mathbb{R}$, the argument in 2 above exhibits a non-archimedean totally ordered group generated by two copies of A; this group is a homomorphic image of $A \coprod A$, which shows that the latter is not hyper-archimedean. ■

Corollary 3.2. *Suppose $\mathfrak{U}$ is a class of hyper-archimedean ℓ-groups which is closed under coproducts and images under ℓ-homomorphisms which preserve the designated unit. Then for each ℓ-group G in $\mathfrak{U}$ and each prime convex ℓ-subgroup M of G, G/M is a rank 1 group.*

We record a few observations, mostly for later use, about the coproduct with a group of rank 1. These are independently interesting in any event.

Remark 3.3. Let A denote a unital subgroup of the rationals. Let us view the application $\gamma^A(G) = G \coprod A$, as a functor on $\ell\mathfrak{Ab}^*$:

1. Suppose that G is a unital totally ordered group, and let $d : G \longrightarrow dG$ and $d' : A \longrightarrow dG$ denote, respectively, the natural embedding of G in its divisible hull dG and the unique unital morphism of A into dG, which is also an embedding.

   The reader will observe that $G \coprod A$ *is the unital ℓ-subgroup of dG generated by the images of G under d and A under d'.*

   Proof. For simplicity, let us identify G and A with their respective images. Suppose now that $f : G \longrightarrow B$ and $f' : A \longrightarrow B$ are two unital morphisms; since $G \cap A$ is a rank 1 group, f and f' agree on it. This implies that the function f^* defined on $G + A$ by
   $$f^*(g + a) = f(g) + f'(a),$$
   is well defined and a homomorphism. It is also easy to show that it preserves order, and so f^* is a unital map. ■

2. Since $A \coprod A = A$, it follows that γ^A is a reflection. The subcategory in which it reflects consists of all the unital ℓ-groups G for which $A1 \subseteq G$. Apart from this description, one should observe that γ^A is a monoreflection, as the coprojection $G \longrightarrow G \coprod A$ is one-to-one. Otherwise, there is very little to say, in general, except to give the following cautionary example.

3. It is tempting to conclude that $G \coprod \mathbb{Q}$ is the divisible hull of G, but this is not the case. Let $G = \mathbb{Z}[\sqrt{2}]$; then $G \coprod \mathbb{Q} = \mathbb{Q} + \mathbb{Z}\sqrt{2}$.

We shall return to consideration of γ^A in the context of groups of step functions.

A variation on one of the arguments in the proof of Theorem 3.1 improves upon Corollary 3.2.

Theorem 3.4. *Suppose $\mathfrak{T}$ is an FTC in $\ell\mathfrak{Ab}^*$ consisting of archimedean ℓ-groups. Then, for each $G \in \mathfrak{T}$, G in its Yosida representation is an ℓ-group of step functions.*

Proof. Suppose that $G \in \mathfrak{T}$, and assume that $G \subseteq C(Y)$ in its Yosida representation does not consist of step functions, and let $0 < g \in G$ be a function which has infinite range. Note that, by Corollary 3.2, the functions in G are rational-valued.

Without loss of generality, we may assume there is a sequence of points of Y, $p_1, p_2, \ldots$, such that the sequence $(g(p_n))_n$ is increasing and converges to t_0. In the following, let $t_n = g(p_n)$. We go through a number of reductions to derive the desired contradiction.

1. t_0 *is irrational.* Otherwise, scaling by the appropriate integer, we may assume t_0 is an integer. Then the function $g - t_0 1$ is not bounded away from zero, which cannot be if G is hyper-archimedean.

2. Restrict to $P = \{ p_1, p_2, \ldots \}$, and denote the group of the restrictions of functions in G to P by H. Since H is a homomorphic image of G in the category $\ell\mathfrak{Ab}^*$, we have $H \in \mathfrak{T}$.

3. Note that in H we have the constant function 1 and the sequence t defined above. Let A denote the subgroup of H generated by 1 and t; the reader will easily check that A is an ℓ-subgroup and an $\ell\mathfrak{Ab}^*$-subobject of H, and that it consists of the sequences that eventually integral combinations of 1 and t. Since FTCs are closed under formation of subobjects, $A \in \mathfrak{T}$.

4. Finally, we exhibit a unital ℓ-group B which is nonarchimedean, yet generated by two copies of A, and therefore an image under a $\ell\mathfrak{Ab}^*$-morphism of $A \coprod A$. This implies that $A \coprod A$ fails to be hyper-archimedean, contrary to assumptions.

   Observe that each $a \in A$ converges to $m + nt_0$, where $a_k = m + nt_k$, for all but finitely many k. Now $B = A \times \mathbb{Z}t_0$, ordered as follows: $(a, nt_o) > 0$ if a is positive (pointwise) and converges to a strictly positive number, or else a is eventually zero and then $nt_0 > 0$.

   Next, map $A \longrightarrow B$ by u_1 and u_2 defined by $u_1(a) = (a, nt_0)$ and $u_2(a) = (a, -nt_o)$, with a converging to $m + nt_0$. As in the proof of Theorem 3.1, one easily verifies that these are maps of $\ell\mathfrak{Ab}^*$. These two maps embed A as distinct copies which together generate B. Thus, there is a surjective $u : A \coprod A \longrightarrow B$ in $\ell\mathfrak{Ab}^*$, extending u_1 and u_2, as promised.

As explained before, this completes the proof of this theorem. ■

4 FTCs of Archimedean ℓ-Groups: Step Functions.

We identify a large family of FTCs consisting of archimedean ℓ-groups, one for each unital subgroup of $\mathbb{Q}$. There are continuously many of them. We formalize the conversation about step functions to have a smoother exposition.

Definition & Remarks 4.1. Let G be an $\ell\mathfrak{Ab}^*$-object. Call $e \in G$ a *component* if $e \wedge (1-e) = 0$.

(a) Suppose A denotes a unital subgroup of $\mathbb{Q}$. A finite combination $r_1e_1 + \cdots + r_me_m$, with $r_i \in A$ and each e_i a component, is an *A-step function.* Any step function may easily be rewritten such that $e_i \wedge e_j = 0$, for distinct i and j. In such a representation we call each r_ie_i an *r_i-step.*

Let $\mathfrak{S}(A)$ denote the subcategory of $\ell\mathfrak{Ab}^*$ consisting of all groups of A-step functions.

(b) We return to the functor γ^A (with A a unital subgroup of $\mathbb{Q}$) and the discussion in 3.3. To recall, $\gamma^A(G) = G \coprod A$, for any unital ℓ-group G. We consider now its restriction to $\mathfrak{S}(\mathbb{Q})$.

In order to formulate statements about γ^A sensibly, we introduce the following: suppose that G is a unital ℓ-group of step functions. If $Af \in G$, for each component f of the designated unit, we will call G an *A-vector lattice.*

Note that if A is a subring of $\mathbb{Q}$, then stipulating that G is an A-vector lattice simply means that G is an A-module in which $0 \leq r \in A$ and $0 \leq g \in G$ imply that $rg \geq 0$.

Recall that when a unital subgroup A of $\mathbb{Q}$ has a unital morphism to an $\ell\mathfrak{Ab}^*$-object H it is unique (Theorem 3.1). There can, therefore, be no confusion in writing $A \leq H$ when such a unital map exists.

We then have the following observation. The proof is straightforward, and is left to the reader.

Lemma 4.2. *If $G \in \mathfrak{S}(B)$ and an A-vector lattice, then $A \subseteq B$, Moreover, the following are equivalent:*

(a) *$A \leq G$.*

(b) *$r1$ is defined, for each $r \in A$.*

(c) *G is an A-vector lattice.*

In the above lemma, the A and B need not coincide:

Example 4.3. Let G be the group of all sequence of rational numbers which are eventually an integer constant. Then $G \in \mathfrak{S}(\mathbb{Q})$, but merely a $\mathbb{Z}$-vector lattice.

In preparation for Proposition 4.4, the following comments are very helpful.

For each unital ℓ-homomorphism $h : G \longrightarrow H$ between members of $\mathfrak{S}(B)$ there a boolean homomorphism $\mathcal{E}(h) : \mathcal{E}(G) \longrightarrow \mathcal{E}(H)$ between the boolean algebras of components of G and H. If G is an A-vector lattice, then so is H, and $h(re) = r\mathcal{E}(h)(e)$, for each $r \in A$ and $e \in \mathcal{E}(G)$. This means that h can be reconstructed from a boolean homomorphism φ of the algebras of components, by defining f via the formula $f(re) = r\varphi(e)$.

With a straightforward, albeit tedious calculation one can then prove the following extension of 3.3.1.

For convenience let us agree now that if G is any unital ℓ-group, and A any unital subgroup of $\mathbb{Q}$, then $\chi_A(G)$ denotes the subset of G of all the A-step functions. It is clearly a unital ℓ-subgroup of G.

Proposition 4.4. *Suppose that G is a unital ℓ-group of B-step functions and an A-vector lattice, with $A \subseteq B$, and both unital subgroups of $\mathbb{Q}$.*

Then, for each unital subgroup E of $\mathbb{Q}$, $G \coprod E$ is the ℓ-subgroup of dG generated by G and E, and it is an $(A + E)$-vector lattice in $\mathfrak{S}(B + E)$. In particular, we have the following – taking $E = B$:

(a) *γ^B is a monoreflection of $\mathfrak{S}(B)$ in the full subcategory $\mathfrak{S}_v(B)$ of its B-vector lattices.*

(b) *γ^B induces an equivalence of categories, from $\mathfrak{S}(\mathbb{Z})$ to $\mathfrak{S}_v(B)$.*

Proof. We begin by describing the unital ℓ-subgroup S generated by G and E in dG. The reader will easily verify that S consists of all expressions

$$g = \sum_{i=1}^{n} (r_i + s_i)e_i,$$

with each r_i an A-value of G, $s_i \in E$, and each $e_i \in \mathcal{E}(G)$. Without loss of generality, one may take the e_i to be pairwise disjoint, and it is clear that such an expression is unique.

Denote the inclusions of G and E in S by u_G and u_E, respectively. Then, since there is at most one unital morphism out of E, whenever v_G and v_E are unital maps from G and E into H, it suffices to show that there is a unique unital map $v : S \longrightarrow H$ such that $v \cdot u_G = v_G$.

To that end, let $\theta : \mathcal{E}(G) \longrightarrow \mathcal{E}(H)$ be the map induced by v_G on the boolean algebra of components of G. Define

$$v\Big(\sum_{i=1}^{n} (r_i + s_i)e_i\Big) = \sum_{i=1}^{n} (r_i + s_i)\theta(e_i).$$

It is easy, but hardly illuminating, to check that v is a homomorphism, and since it assigns disjoint pairs to disjoint pairs, it is, in fact an ℓ-homomorphism, and obviously unital. Finally, taking all the $s_i = 0$ in the definition of v, one recovers v_G; that is, $v \cdot u_G = v_G$, as promised.

With $B = E$, (a) is clear, as γ^B is obviously idempotent. As for (b), observe that the coreflection $\chi_{\mathbb{Z}}$ inverts γ^B. ■

With this result in hand, we shall be able to tell the full story around the upcoming diagram (4.6.1).

Next, we have the theorem which establishes, among other things, that there are at least continuously many FTCs of archimedean ℓ-groups.

Theorem 4.5. (a) *Each $\mathfrak{S}(A)$ is an FTC in $\ell\mathfrak{Ab}^*$.*

(b) *$\mathfrak{S}(\mathbb{Z})$ is a minimal nontrivial FTC and, $\mathfrak{S}(\mathbb{Q})$ the largest consisting of archimedean ℓ-groups.*

(c) *If $\mathbb{Z} \subseteq A_1, A_2 \subseteq \mathbb{Q}$ are distinct, then $\mathfrak{S}(A_1) \neq \mathfrak{S}(A_2)$.*

Proof. (a) It is easily checked that the image under any $\ell\mathfrak{Ab}^*$-map of a component is a component or zero. Thus, it is also easy to see that the image of an A-step function is an A-step function, proving $\mathfrak{S}(A)$ is closed under quotients.

If G is any $\ell\mathfrak{Ab}^*$-object, then it is easily checked that if $h : A \longrightarrow G$ is a unital ℓ-homomorphism and A is in $\mathfrak{S}(A)$. then $h(A) \subseteq \chi_A(G)$.

Suppose G is any $\mathfrak{S}(A)$-object, and H is an $\ell\mathfrak{Ab}^*$-subobject. Let $a = r_1e_1 + \cdots + r_me_m \in H$, with $r_i \in A$, each e_i a component, and $e_i \wedge e_j = 0$ $(i \neq j)$, and in addition, assume without any loss of generality $0 \leq r_1 < \cdots < r_m$. The claim is that each $e_i \in H$; the proof is by induction on the number of components.

Scaling appropriately, we may, without loss of generality, take each $r_i \in \mathbb{N}$. One checks easily:

$$b = (a - r_{m-1}1) \vee 0 = (r_m - r_{m-1})e_m \in H,$$

Letting $k + 1 = r_m - r_{m-1}$, we get

$$(b - k1)e_m \vee 0 = e_m \in H.$$

Thus, each $e_i \in H$, and so $H \in \mathfrak{S}(A)$. This suffices.

(b) Since in any FTC the objects are represented as rational-valued step functions, $\mathfrak{S}(\mathbb{Q})$ is the largest FTC of archimedean ℓ-groups.

If $\mathfrak{T} \subseteq \mathfrak{S}(\mathbb{Z})$ is a nontrivial FTC, it has a copy of $\mathbb{Z} \boxplus \mathbb{Z}$; note that every unital abelian ℓ-group generated by one element is a quotient of this group, and, in particular, so is every $\mathfrak{S}(\mathbb{Z})$-object with one generator. Taking coproducts in $\ell\mathfrak{Ab}^*$ of copies of $\mathbb{Z} \boxplus \mathbb{Z}$, and then all quotients of those coproducts, one gets $\mathfrak{T} = \mathfrak{S}(\mathbb{Z})$.

(c) is obvious. ■

We preface the conclusion of this section with an example, some comments, and with a diagram of the FTCs of archimedean ℓ-groups we know. In the sequel, A denotes a fixed subgroup of the reals.

Remark 4.6. Let $\mathbb{T}(\mathcal{A})$ denote the least FTC containing the class $\mathcal{A}$. Note that $\mathbb{T}(A)$ consists of the set of unital subgroups of A and $\{0\}$.

(a) Suppose that $\mathfrak{T}$ is an FTC and that the set $\{\, E_i \,:\, i \in I \,\}$ is a family of subgroups of $\mathbb{Q}$ all of which are members of $\mathfrak{T}$. As in preceding arguments, it can easily be checked that their coproduct is $E = \sum_i E_i$, the subgroup generated by the E_i. In turn, $\mathfrak{T}$ contains all the unital subgroups of E; that is $\mathbb{T}(E) \subseteq \mathfrak{T}$. We state explicitly:

> *If $\mathfrak{T}$ is any FTC in $\ell\mathfrak{Ab}^*$, then there is a largest rank 1 subgroup E of $\mathbb{R}$, such that $\mathbb{T}(E) \subseteq \mathfrak{T}$.*

We will say that $\mathfrak{T}$ is a *rank E FTC.*

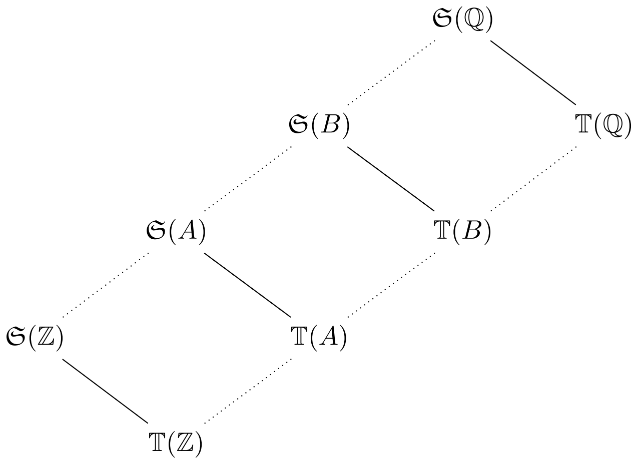
(b) Note that if $\mathfrak{T}$ consists of archimedean groups, and has rank $\mathbb{Q}$, then by Theorem 4.5(b), $\mathfrak{T} = \mathfrak{S}(\mathbb{Q})$.

(c) If the rank of $\mathfrak{T}$ is $\mathbb{Z}$ we say that $\mathfrak{T}$ has *integral rank.* Regarding integral rank, and rank in general, we have Theorem 4.7 and its corollary, which conclude the section.

(d) Meanwhile, here is the promised diagram of known FTCs of archimedean ℓ-groups. It is the outline of a lattice, of which $\mathfrak{S}(\mathbb{Q})$ is the largest element, and $\mathbb{T}(\mathbb{Z})$ is the least. The dotted lines connecting the $\mathfrak{S}(E)$s and also the ones linking the $\mathbb{T}(E)$s indicate that there are (continuously) many FTCs in between.

It is assumed in (4.6.1) that A is a proper unital subgroup of B.

(4.6.1)



Note that (with $A \subseteq B$), $\mathbb{T}(B) \cap \mathfrak{S}(A) = \mathbb{T}(A)$. On the other hand, if G consists of B-step functions, then (by the argument of the proof of Proposition 4.4), $G \subseteq$

$\gamma^B(\chi_A(G))$, and the latter is a B-vector lattice, belonging to the FTC $\mathfrak{S}(A) \bigvee \mathbb{T}(B)$, and so $\mathfrak{S}(B) = \mathfrak{S}(A) \bigvee \mathbb{T}(B)$.

We can now describe all the FTCs in $\ell\mathfrak{Ab}^*$ that consist of archimedean ℓ-groups.

Theorem 4.7. *Suppose that $\mathfrak{T}$ is an FTC consisting of archimedean ℓ-groups. Let A be the subgroup of $\mathbb{Q}$ generated by all $r \in \mathbb{Q}$ for which there is a $G \in \mathfrak{T}$ and a step function $f \in G$ with an r-step.*

Then $\mathfrak{T}$ has rank A, and $\mathbb{T}(A) \subseteq \mathfrak{T} \subseteq \mathfrak{S}(A)$. Moreover, if $\mathfrak{T}$ contains a group that is not totally ordered, then $\mathfrak{T} = \mathfrak{S}(A)$.

Proof. Let $f \in G$, with $G \in \mathfrak{T}$, having an r-step re, where e is a component of 1. Represent G as a group of step functions on the Stone dual X of the boolean algebra of components. Observe that, for each $x \in X$ where $e(x) = 1$, $f(x) = r$, proving that $\mathbb{Z}r \in \mathfrak{T}$ and that the rank of $\mathfrak{T}$ contains A. Since the converse is obvious, we have that A is the rank of $\mathfrak{T}$.

Next, if $\mathfrak{T}$ contains a group which is not totally ordered, then by an earlier argument, $\mathbb{Z} \boxplus \mathbb{Z} \in \mathfrak{T}$ and $\mathfrak{S}(\mathbb{Z}) \subseteq \mathfrak{T}$, whence $\mathfrak{T} = \mathfrak{S}(A) = \mathfrak{S}(\mathbb{Z}) \vee \mathbb{T}(A)$, by the comments in 4.6(d). ■

Together with Theorem 4.5, the preceding result has the following immediate corollary.

Corollary 4.8. (a) *The only nontrivial FTC of archimedean ℓ-groups of integral rank is $\mathfrak{S}(\mathbb{Z})$.*

(b) *If $\mathfrak{T}$ is an FTC of archimedean unital ℓ-groups, then either $\mathfrak{T} = \mathbb{T}(A)$, for a suitable A unital subgroups of $\mathbb{Q}$, or else $\mathfrak{T} = \mathfrak{S}(A)$.*

5 FTCs with Nonarchimedean ℓ-Groups.

We have already seen that for each unital subgroup A of $\mathbb{R}$, of rank at least 2, $\mathbb{T}(A)$ contains nonarchimedean groups. In this section we shall describe $\mathbb{T}(A)$ more precisely. Throughout the section lexicographic extensions will figure prominently, and it therefore makes sense to stipulate the following convention: for any totally ordered group A and any ℓ-group G, $A \overset{\rightarrow}{\times} G$ denotes the direct product with lexicographical, ordering from left to right.

Let us begin by stating a very simple lemma. The reader should recall that any finitely generated torsion free abelian group is free.

Lemma 5.1. *If an FTC $\mathfrak{T}$ contains the finitely generated unital subgoup $A \subseteq \mathbb{R}$ of rank at least 2, then $A \overset{\rightarrow}{\times} (\mathbb{Z} \boxplus \mathbb{Z}) \in \mathfrak{T}$.*

Proof. Without losing any generality, assume $A = \mathbb{Z} \oplus \mathbb{Z}r$, for a suitable irrational r. Let $G = A \overset{\rightarrow}{\times} (\mathbb{Z} \boxplus \mathbb{Z})$, and define $t_1, t_2 : A \longrightarrow G$ by $t_1(m + nr) = (m + nr, n, 0)$ and $t_2(m + nr) = (m + nr, 0, n)$. It is easily checked that t_1 and t_2 extend to a unital quotient map $t : A \coprod A \longrightarrow G$. ■

Taking into account that each unital subgroup A of $\mathbb{R}$ is an updirected union of finitely generated ones, and that FTCs are closed under the formation of direct limits, one can remove the "finitely generated" provision from the preceding lemma.

Lemma 5.2. *If an FTC $\mathfrak{T}$ contains the unital subgoup $A \subseteq \mathbb{R}$ of rank at least 2, then $A \overset{\rightarrow}{\times} (\mathbb{Z} \boxplus \mathbb{Z}) \in \mathfrak{T}$.*

The mild surprise at this point is that $\ell\mathfrak{Ab}$, the category of all abelian ℓ-groups and all ℓ-homomorphisms, enters the conversation.

Lemma 5.3. *Assume that A is a unital subgroup of $\mathbb{R}$. Suppose $G_1, G_2, \ldots$ are arbitrary ℓ-groups, and G is their coproduct in $\ell\mathfrak{Ab}$. Suppose, further, that each $A \overset{\rightarrow}{\times} G_i$ is a unital quotient of a unital coproduct of copies of A. Then $A \overset{\rightarrow}{\times} G$ is a unital quotient of a unital coproduct of copies of A.*

Proof. Let $u_i : G_i \longrightarrow G$ denote the i-th coprojection, and

$$f_i = A \overset{\rightarrow}{\times} u_i : A \overset{\rightarrow}{\times} G_i \longrightarrow A \overset{\rightarrow}{\times} G$$

be the map defined by $f_i(a, x) = (a, u_i(x))$. Note that each f_i is a unital morphism.

We were given unital quotients q_i from various iterated unital coproducts of copies of A onto $A \overset{\rightarrow}{\times} G_i$. The images of the $f_i \cdot q_i$ generate $A \overset{\rightarrow}{\times} G$, which implies that the unital morphism h (out of a unital coproduct of copies of A) induced by the $f_i \cdot q_i$ is surjective. ■

Theorem 5.4. *Suppose an FTC $\mathfrak{T}$ contains a unital subgroup $A \subseteq \mathbb{R}$ of rank ≥ 2. Then $A \overset{\rightarrow}{\times} H \in \mathfrak{T}$, for each abelian ℓ-group H.*

Proof. By Lemmas 5.1 and 5.3, $A \overset{\rightarrow}{\times} F \in \mathfrak{T}$, for each free abelian ℓ-group F. Since each abelian ℓ-group is an ℓ-homomorphic image of a free one, and whenever H is an ℓ-homomorphic image of F in $\ell\mathfrak{Ab}$, $A \overset{\rightarrow}{\times} H$ is obviously an ℓ-homomorphic image of $A \overset{\rightarrow}{\times} F$ in $\ell\mathfrak{Ab}^*$, this suffices to complete the proof. ■

Concerning the properties of unital subgroups of $\mathbb{R}$, the distinction of rank one groups once again surfaces.

Definition & Remarks 5.5. Throughout this commentary, A denotes a unital subgroup of $\mathbb{R}$. Let $A \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ denote the full subcategory of $\ell\mathfrak{Ab}^*$ the objects of which are of the form $A \overset{\rightarrow}{\times} G$, where G is an abelian ℓ-group. Without any indication to the contrary, it is assumed that the distinguished unit of $A \overset{\rightarrow}{\times} G$ is $(1, 0)$.

If $h : A \overset{\rightarrow}{\times} G \longrightarrow A \overset{\rightarrow}{\times} G'$ is any unital map, then it is easy to see that, for each $x \in G$, $h(0, x) = (0, y)$, for some $y \in G'$. That is to say, h induces an ℓ-homomorphism $\delta_A(h) : G \longrightarrow G'$. As we shall see presently, it would be useful if δ_A were the unique ℓ-homomorphism such that

$$(\dagger) \qquad h(r, g) = (r, \delta_A(h)(g)), \quad r \in A, \quad g \in G.$$

This is the case provided A is a subgroup of $\mathbb{Q}$, because, if this is the case, then – since $h(m, 0) = (m, 0)$, for every integer m – the same is true for every rational.

As the following example demonstrates, (†) fails in general, and, for divisible groups, always, when A contains an irrational.

Example 5.6. Let A be a divisible unital subgroup of $\mathbb{R}$, containing the irrational number r. We may write $A = \mathbb{Q}r \oplus B$, for a suitable divisible unital subgroup B of $\mathbb{R}$. Note that each member of A may be expressed uniquely as $qr + b$, with $q \in \mathbb{Q}$ and $b \in B$.

Consider $G = A \overset{\rightarrow}{\times} \mathbb{Q}$ and let $f : G \longrightarrow G$ be the map defined by

$$f(qr + b, t) = (qr + b, q + t), \quad q, t \in \mathbb{Q}, \quad b \in B.$$

It is easy to check that f is a unital order preserving map, for which δ_A is the identity map. However, observe that $f(r, 0) = (r, 1)$, which shows that (†) fails.

We mention the following observation, as it is of some independent interest, but we omit the proof since the result won't be used anywhere, and – for $A = \mathbb{Z}$, at least – it is known to MV-algebraists, in a slightly different formulation, and we shall describe it in 6.2.

Proposition 5.7. *Suppose that A is a unital subgroup of $\mathbb{R}$ satisfying (†). Then $A \overset{\rightarrow}{\times} (\cdot)$ defines an equivalence of the category $\ell\mathfrak{Ab}$ onto the full subcategory $A \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ of $\ell\mathfrak{Ab}^*$. The inverse equivalence is γ^A.*

Finally, in this section, we have a result along the lines of Theorem 5.4, with an example to follow. We use the customary notation $0 \leq a \ll b$ in an ℓ-group to signify that every multiple of a is bounded above by b.

Theorem 5.8. *$\mathbb{Z} \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ is the least FTC containing a nonarchimedean unital ℓ-group.*

Proof. It is easy to show that $\mathbb{Z} \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ is closed under formation of unital ℓ-homomorphic images. for if $f : G = \mathbb{Z} \overset{\rightarrow}{\times} A \longrightarrow G'$ is such a surjective map, then, letting $Z = f(\{\,(n, 0) : n \in \mathbb{Z}\,\})$ and $A' = f(\{\,(0, g) : g \in A\,\})$, we have, first, that $Z \cong \mathbb{Z}$, since the map is unital, and $G' = Z \oplus A'$, with $x \ll z$, for each $x \in A'$ and $z \in Z$. This shows that $G' = Z \overset{\rightarrow}{\times} A$.

Next, let us identify the coreflection in a given unital ℓ-group G. Let A denote $\{\,x \in G : x \ll 1\,\}$. If Z stands for the subgroup of G generated by 1, then one verifies without any trouble that $A \cap Z = \{0\}$ and that $Z + A = Z \overset{\rightarrow}{\times} A$, and that this is an ℓ-subgroup. Denote $Z \overset{\rightarrow}{\times} A = G_{\mathbb{Z}}$. To show that $G_{\mathbb{Z}}$ is the desired coreflection, it suffices to establish that every unital ℓ-subgroup $H = \mathbb{Z} \overset{\rightarrow}{\times} B$ of G is, in fact, contained in $G_{\mathbb{Z}}$. This is obvious, and we leave the verification to the reader.

To conclude the proof that $\mathbb{Z} \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ is an FTC, suppose that H is a unital ℓ-subgroup of $\mathbb{Z} \overset{\rightarrow}{\times} C$, and $C' = \{\,y \in C : (0, y) \in H\,\}$; then $H = \mathbb{Z} \overset{\rightarrow}{\times} C'$.

Finally, if $\mathfrak{T}$ is any FTC containing a nonarchimedean ℓ-group K, then $0 < x \ll 1$, for a suitable $x \in K$. Thus $\mathfrak{T}$ contains a copy of $\mathbb{Z} \overset{\rightarrow}{\times} \mathbb{Z}$, and, taking coproducts, a copy of $\mathbb{Z} \overset{\rightarrow}{\times} (\mathbb{Z} \boxplus \mathbb{Z})$. Arguing as in the proof of Theorem 5.4 by iterating coproducts, $\mathfrak{T}$ contains every $\mathbb{Z} \overset{\rightarrow}{\times} A$. ■

Remark 5.9. Suppose that A is a unital subgroup of $\mathbb{Q}$. Let $(\downarrow A) \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ denote the class of all $B \overset{\rightarrow}{\times} H$, with B a unital subgroup of A, and H any abelian ℓ-group. As in the proof of Theorem 5.8, one shows that, for each unital subgroup A of the rationals, $(\downarrow A) \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ is monocoreflective.

However, $(\downarrow A) \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ need not be an FTC. The subgroup H of $\mathbb{Q} \overset{\rightarrow}{\times} \mathbb{Q}$ generated by $(\frac{1}{2}, \frac{1}{2})$ and the designated unit $(1, 0)$ is a unital subgroup, but it is not of the form $B \overset{\rightarrow}{\times} K$, for any unital subgroup B of $\mathbb{Q}$.

6 Closing Remarks

We end this report on functorial torsion classes with a brief discussion of the categories which, in a reasonable sense, are related to $\ell\mathfrak{Ab}^*$.

A study of the situation in $\mathfrak{W}^*$, the category of archimedean ℓ-groups with a designated strong unit (and unital ℓ-homomorphisms), which is being assembled by this author and A.W.Hager, will appear in due course. This investigation relies heavily on the intimate relationship between an object A in $\mathfrak{W}^*$ and its Yosida space, and the representation of A as a group of bounded, real-valued continuous functions on that space.

One is able to make short shrift of the situation with FTCs in $\ell\mathfrak{Ab}$, consisting of all abelian ℓ-groups and all ℓ-homomorphisms, which is what we will do, stressing the categorical features that are responsible for the results.

The essential facts are these:

1. In $\ell\mathfrak{Ab}$, every nontrivial object contains a copy of the additive and totally ordered group $\mathbb{Z}$ of integers.
2. $\mathbb{Z} \boxplus \mathbb{Z}$ is the free object on one generator.
3. $\mathbb{Z} \boxplus \mathbb{Z}$ is an ℓ-homomorphic image of $\mathbb{Z} \coprod \mathbb{Z}$, and, by elementary arguments from universal algebra, every free object is an ℓ-homomorphic image of an iterated coproduct of copies of $\mathbb{Z}$.

The above remarks apply to any variety $\mathfrak{V}$ of ℓ-groups, since any ℓ-group generated by one element is abelian, and this is enough to establish the validity of the following observation.

Proposition 6.1. *In any variety $\mathfrak{V}$ of ℓ-groups, the only nontrivial FTC is $\mathfrak{V}$ itself.*

Our concluding remarks are about MV-algebras, and they freely use the Chang-Mundici equivalence of the categories $\ell\mathfrak{Ab}^*$ and $\mathfrak{MV}$, the category of MV-algebras and MV-morphisms. Our references are to [COM00].

Remark 6.2. The Chang-Mundici correspondence associates with each unital ℓ-group the interval $[0, 1]$; as before, 1 denotes the designated strong order unit.

For the record, an *MV-algebra* is an algebra A of type $(\oplus, \neg, 0)$ in which $\oplus$ is an associative and commutative binary operation, for which 0 is the identity, while (i) $x \oplus (\neg 0) = \neg 0$, for each $x \in A$;

$$\text{(ii)} \qquad \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$

for all x and y, and (iii) $\neg(\neg x) = x$, for each $x \in A$. One then denotes $\neg 0 = 1$.

Under the Chang-Mundici equivalence, $\mathbb{R}$ corresponds to the unit interval $\mathbb{I}$, which is endowed with the truncated operation

$$x \oplus y = 1 \wedge (x + y), \quad \text{for all} \quad x, y \in \mathbb{R},$$

while archimedean ℓ-groups correspond to the *semisimple* MV-algebras; that is to say, the MV-algebras that are subdirect products of MV-subalgebras of the unit interval.

The translation of the results in Sections 3 and 4 to MV-algebras can be summarized as follows:

1. An FTC of semisimple MV-algebras consists of MV-algebras that are made up of rational-valued step functions. These are the locally finite MV-algebras, which are characterized in [CDM04, Theorem 5.1].

2. Let A be any MV-subchain of $\mathbb{Q} \cap \mathbb{I}$. Then an FTC of semisimple MV-algebras is either $\mathbb{T}_{MV}(A)$, consisting of the zero MV-chain and all the MV-subchains of A, or $\mathfrak{S}_{MV}(A)$, which is the class of all the MV-algebras of A-valued step functions. (See Corollary 4.8.)

3. Finally, a comment regarding the significance of Theorem 5.8 for MV-algebras. The reader might refer, for additional detail, to [COM00, Section 7.4] or to [DL94].

   First, let us employ the notation $a \ll b$ to mean that $na \leq b$ in an MV-algebra (relative to $\oplus$). Let $\mathrm{Rad}(A)$ denote the set of all $x \in A$ such that $x \ll 1$. Now call the MV-algebra A *perfect* if for each $x \in A$, either $x \in \mathrm{Rad}(A)$ or $-x \in \mathrm{Rad}(A)$.

   Now, under the Chang-Mundici correspondence the subcategory $\mathbb{Z} \overset{\rightarrow}{\times} \ell\mathfrak{Ab}$ reduces to the subcategory of all perfect MV-algebras. Thus, Theorem 5.8 translates as follows: *The subcategory of perfect MV-algebras is an FTC; it is the least FTC containing a nonsemisimple MV-algebra.*

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