Epicompletion in Frames with Skeletal Maps, IV: Strongly Joinfit Frames

Jorge Martínez

Abstract. Earlier work has shown that there is a monoreflection ψ of the category of compact normal, joinfit frames with skeletal frame maps in the subcategory consisting of strongly projectable frames. This article extends the domain of ψ to the strongly joinfit frames. The saturation nucleus s is a reflection with respect to weakly closed frame maps, in the subcategory of subfit frames. Moreover, $s \cdot \psi = \psi \cdot s$, on compact normal, joinfit frames with skeletal, weakly closed frame maps, and $s \cdot \psi$ is an epireflection, but not a monoreflection, in the subcategory of strongly projectable, regular frames, all of which are epicomplete.

This article builds on work in [MZ08a, MZ08b, M08b]. It is part of an ongoing investigation into epicompletion in various categories of compact frames with skeletal frame homomorphisms. In [MZ08a] it was shown that the absolute of a compact regular frame represents the functorial epicompletion. The epicomplete objects here are the strongly projectable compact regular frames.

The objective, in the long run, being to abstract the work of Conrad in [C71] on essential closures of archimedean ℓ-groups, and Carrera's contribution in [Cr04] on the functorial aspects of the passage to the essential closure, it seemed natural to look for a generalization of the frame-theoretic embedding in the absolute. The main accomplishment of [MZ08b] is the construction, by a pushout, of an epireflection ψ of compact normal, joinfit frames in the subcategory whose objects are strongly projectable, and that ψ extends the absolute. In [M08b] it is shown that ψ is a monoreflection, and that the reflection map ψ_A is a closed (skeletal) frame embedding, for each compact normal, joinfit frame A. Indeed, ψ_A is the minimal closed skeletal frame embedding into a strongly projectable frame with the same features. It is also shown in [M08b] that, with coherent skeletal frame maps, the strongly projectable compact regular frames are epicomplete in coherent normal, joinfit frames, and epireflective, but not monoreflective.

In the present paper several aspects of the abovecited articles are improved upon. After review of some pertinent background material, we introduce the strongly joinfit frames: these are the frames which contain their regular coreflection $*$-densely. Strong joinfitness is a natural generalization of the paired hypotheses – namely that all frames be normal and joinfit – that underpin the development of ψ in [MZ08b, M08b]. We sketch in §2 what must be modified in [MZ08b] to extend the domain of ψ to strongly joinfit frames. In the process we show that the awkwardly assembled functor φ of

[M08a] extends to a monocoreflection of frames with skeletal maps in the subcategory of strongly joinfit frames (Theorem 2.8).

Next, we consider the saturation nucleus s as a reflection of compact frames, with "suitable" frame homomorphisms, in the subcategory of subfit frames. These suitable maps include all closed maps, but also those frame maps h that satisfy the more general

$$h(a) \vee u = 1 \implies a \vee h_*(u) = 1.$$

These are the *weakly closed* maps; saturation itself is weakly closed, but not closed. On compact normal, joinfit frames, with weakly closed, skeletal maps, s and ψ commute (Theorem 3.9), and their composite epireflects in the subcategory of strongly projectable regular frames. In this setting – as in that of [M08b] – each strongly projectable regular frame is epicomplete. However, because s is always a retraction on compact normal, joinfit frames, the strongly projectable regular frames do not form a monoreflective subcategory (Corollary 3.12).

The paper concludes with some examples using frames of radical ideals of commutative rings. As was the case at the end of [M08b], we find the question of whether there is a functorial epicompletion on any of the categories of compact joinfit frames considered here (with skeletal maps endowed, perhaps, with additional properties), still steeped in mystery.

1 Preliminaries.

We begin with a list of basic definitions, which the knowledgeable reader ought to be able to skip entirely. For additional information one may refer to [J82] and Chapter 2 of [PT01]. It is assumed that the reader is familiar with basic category theory; our standard reference remains [HS79]. It also seems reasonable to suppose that the reader of this article is familiar with [MZ08a, MZ08b, M08b], as well as [M08a], so that our review of them will be brief. In particular, we feel free to assume that the reader is familiar with algebraic lattices.

Definition 1.1. Throughout, L is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in L$, denote the set of elements of L less than or equal to (resp. greater than or equal to) x by $\downarrow x$ (resp. $\uparrow x$). We denote by $\mathfrak{k}(L)$ the set of sll compact elements of L.

- The algebraic lattice L has the *finite intersection property* (abbr. *FIP*) if for any pair $a, b \in \mathfrak{k}(L)$, $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema. L is *coherent* if 1 is compact and L has the FIP.
- The *Heyting operation* $a \to b$ (in a frame L):
$$a \to b = \vee \{ x \in L \,:\, a \wedge x \leq b \}.$$
Also put $x^{\perp} \equiv x \to 0$.

- $p \in L$ a *polar*: one of the form $p = y^{\perp}$, for some $y \in L$. It is well known that the set $\mathcal{P}L$ of all polars forms a complete boolean algebra, in which infima agree with those in L.
- In a frame, a is *well below* b, written $a \preceq b$: $b \vee a^{\perp} = 1$.
- $x \in L$ is *regular*: $x = \vee\{\, a \in L \,:\, a \preceq x \,\}$. Let $\mathrm{Reg}(L)$ denote the subset of all regular elements of L. A frame is L *regular*: each element of L is regular.
- L is *normal*: whenever $x \vee y = 1$, there exist disjoint $u \wedge v = 0$ in L, such that $u \leq x$ and $v \leq y$, and $1 = x \vee v = u \vee y$.
- Let j be a closure operator j on a frame L.
  - j is *dense*: $j(0) = 0$.
  - $jL \equiv \{\, x \in L \,:\, j(x) = x \,\}$. Note that j is dense if and only if $0 \in jL$.
  - j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$. It is well known that j is a nucleus $\Longleftrightarrow$ in jL: $x \in jL$ implies that $a \rightarrow x \in jL$, for each $a \in L$.

We record a brief comment concerning frame homomorphisms and their adjoints.

Definition & Remarks 1.2. We start in the category $\mathfrak{Frm}$ of all frames and all frame homomorphisms. If $h : L \longrightarrow M$ is a $\mathfrak{Frm}$-morphism, then $h_* : M \longrightarrow L$ denotes its right adjoint; that is, the map defined by

$$x \leq h_*(y) \iff h(x) \leq y, \quad \text{for all} \quad x \in L,\ y \in M.$$

The following are well known:

1. h_* preserves all infima.
2. $x \leq h_* \cdot h(x)$, for each $x \in L$, and $h \cdot h_*(y) \leq y$, for each $y \in M$. Thus, $h \cdot h_* \cdot h = h$ and $h_* \cdot h \cdot h_* = h_*$.
3. h is *dense* if $h(x) = 0$ implies $x = 0$. Then the following are equivalent: (i) h is dense; (ii) j is dense; (iii) $h_*(0) = 0$.

It follows from the above properties that h is one-to-one if and only if $h_* \cdot h = 1_L$, and that h is surjective if and only if $h \cdot h_* = 1_M$.

We say that h is $*$-*dense* if $h_*(y) = 0$ implies that $y = 0$. Note that this follows if h is surjective, because h_* is one-to-one.

Next, we briefly review two special types of frame homomorphisms: skeletal maps and closed maps, both modelled after notions in topology.

Definition & Remarks 1.3. (a) The frame homomorphism $h : L \longrightarrow M$ is *skeletal* if $x^{\perp\perp} = 1$ in L implies that $h(x)^{\perp\perp} = 1$. It is easy to verify that h is skeletal if and only if

$$x_1^{\perp} = x_2^{\perp} \implies h(x_1)^{\perp} = h(x_2)^{\perp}.$$

Then it is also easy to see that h is skeletal precisely when there is a (unique) frame homomorphism $\mathcal{P}(h) : \mathcal{P}L \longrightarrow \mathcal{P}M$ making the diagram below commute:

(1.3.1)

$$\begin{array}{ccc} L & \xrightarrow{\ h\ } & M \\ \downarrow{\scriptstyle p_L} & & \downarrow{\scriptstyle p_M} \\ \mathcal{P}L & \xrightarrow{\ \mathcal{P}(h)\ } & \mathcal{P}M \end{array}$$

In Figure (1.3.1), p_L denotes the nucleus defined by $p_L(x) = x^{\perp\perp}$. (We do not decorate the $\perp$s to indicate which frame the complements are taken in.)

Applied to $\mathfrak{FrmS}$, the category of frames with skeletal maps, $\mathcal{P}$ may be regarded as an epireflection in the full subcategory of boolean frames ([HS79, BaP96]).

(b) A map of the form $x \mapsto x \vee a$ from A onto $\uparrow a$ is a *closed quotient.* Next, suppose that $h : A \longrightarrow B$ is a frame homomorphism, and let $q : B \longrightarrow F$ be a frame surjection. Factor $q \cdot h = m \cdot e$ through the image, as indicated in the square below:

(1.3.2)

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & B \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle q} \\ E & \xrightarrow{\ m\ } & F \end{array}$$

We say that h is *closed* if for each closed quotient q, e too is a closed quotient.

It is shown in [PT01, Chapter II, §5] that the following are equivalent:

- $h : A \longrightarrow B$ is closed.
- $h_*(h(a) \vee y) = a \vee h_*(y)$, for each $a \in A$ and $y \in B$.
- $h(b) \leq h(a) \vee y \implies b \leq a \vee h_*(y)$.

It is well known that if $h : A \longrightarrow B$ is a frame surjection, A is regular, and B is compact, then h is closed, and hence a closed quotient ([J82, Chapter III, Proposition 1.2]). Using this fact, and the property that a frame-homomorphic image of a regular frame is regular, it is easy to prove that every frame embedding $m : A \longrightarrow B$ of compact regular frames is closed.

We include an account of our conventions in naming categories.

Remark 1.4. $\mathfrak{Frm}$ stands for the parent category (of frames and frame homomorphisms).

- A $\mathfrak{K}$ (generally up front) indicates that the frames in the category question are compact.
- $\mathfrak{Reg}$, $\mathfrak{N}$ and $\mathfrak{Ar}$ denote that the frames are, respectively, regular, normal and joinfit.
- $\mathfrak{Al}$ and $\mathfrak{Ch}$ in the name signal that the frames in the category are, respectively, algebraic with the FIP and coherent, where it is understood that in both instances – involving categories of algebraic frames – the maps are *coherent*; that is to say, they carry compact elements to compact elements.
- A $\mathfrak{C}$ and an $\mathfrak{S}$ at the end of the name tell the reader that the morphisms of the category in question are closed and skeletal, respectively.
- The prefix $\mathfrak{SP}$ indicates that the members of the category in question are strongly projectable. A frame is *strongly projectable* when every polar is complemented.

Thus, for example, $\mathfrak{KNAlS}$ denotes the category of all compact, normal, algebraic frames with FIP, together with skeletal, coherent maps.

2 Strong Joinfitness and ψ.

2.I. Looking Back. Joinfitness mimicks in frames the semisimplicity in commutative rings. Originally, in [M08a], it is introduced as a device to handle archimedean frames without the resource of “points”. Archimedean lattices first appear in [M73], as a lattice-theoretic abstraction of an archimedean lattice-ordered group.

Here we limit ourselves to the highlights from [M08a], and refer the reader to that article for further elaboration. Our objective is to introduce strong joinfitness, a useful generalization of the coupled properties of normality and joinfitness.

Definition & Remarks 2.1. Let A denote an arbitrary frame.

(a) If for each $0 < a \leq b \in A$, there exists a $c \in A$, with $c < b$, such that $b = a \vee c$, we say that A is *joinfit.* If A is algebraic and this condition holds with a, b and c compact, then we say that A is *finitely* joinfit. Observe as well, that to verify joinfitness it suffices to test the defining condition with $b = 1$.

It is straightforward that a joinfit algebraic frame is finitely joinfit. The converse holds in a compact algebraic normal frame.

(b) $\mathrm{Max}(A)$ denotes the set of all maximal elements of A. A is *archimedean* if, for each $c \in \mathfrak{k}(A)$, $\bigwedge \mathrm{Max}(\downarrow c) = 0$. If A is compact then it is archimedean if and only if $\bigwedge \mathrm{Max}(A) = 0$. Assuming *Choice*, an algebraic frame is archimedean precisely when it is finitely joinfit ([M08a, Proposition 2.3]).

[M08a, Proposition 2.5] establishes an important link between joinfitness and the regular coreflection of a frame. The reader may have to review the basic features of the regular coreflection to proceed; there is an adequate account of it in [M08a]. Denote the regular coreflection of A by ρA; if A is normal, then ρA is the subframe of regular elements of A. We denote the inclusion of ρA in A by ρ (or ρ_A, to avoid confusion).

We sharpen the formulation of [M08a, Proposition 2.5]. The proof in [M08a] suffices for the present claim; it proves more than is stated there.

Proposition 2.2. *Suppose A is any frame. If ρ is $*$-dense, then A is joinfit and, for each $a \in A$,*

$$a^{\perp} = \rho(\rho_*(a))^{\perp}.$$

In particular, if A is joinfit and normal then ρ is $*$-dense.

The $*$-density of ρ_A deserves a name.

Definition 2.3. We shall say that A is *strongly joinfit* if ρ_A is $*$-dense. The designation $\mathfrak{A}^2$ will be used for strongly joinfit frames. The rules established in 1.4 then apply to the labels $\mathfrak{K}\mathfrak{A}^2\mathfrak{S}$ and $\mathfrak{S}\mathfrak{P}\mathfrak{A}^2\mathfrak{S}$.

There are many natural examples of coherent, strongly joinfit frames that are not normal. An example is provded in 4.2.

2.II. **ψ Extended.** We review the construction of the monoreflection ψ and extend it from $\mathfrak{K}\mathfrak{N}\mathfrak{A}\mathfrak{r}\mathfrak{S}$ to $\mathfrak{K}\mathfrak{A}^2\mathfrak{S}$. Enough references to [MZ08a, MZ08b], and [M08b] will be provided to help the reader.

Remarks 2.4. Throughout these remarks A denotes a compact frame.

We outline the steps leading up to the definition of ψ, noting *where the extension to $\mathfrak{K}\mathfrak{A}^2\mathfrak{S}$ makes a difference in the statement or proof (or both).* We will cite the original, formulate the generalization, and comment where there are changes in the proof.

1. ([MZ08b, Proposition 4.4]) *The coproduct of compact strongly joinfit frames is strongly joinfit.*

   Proof. Start with a coproduct $A = \coprod_{\lambda\in\Lambda} A_\lambda$, and assume that each A_λ is compact strongly joinfit. Let $R_\lambda = \rho A_\lambda$ and $\rho_\lambda = \rho_{A_\lambda} : R_\lambda \longrightarrow A_\lambda$. Further, denote $R = \coprod_{\lambda\in\Lambda} R_\lambda$, and let $r : R \longrightarrow A$ be the frame map induced by the ρ_λ; that is

   $$r(\otimes_\Lambda x_\lambda) = \otimes_\Lambda \rho_\lambda(x_\lambda).$$

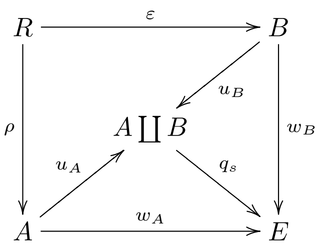
Since each R_λ is $*$-dense in A_λ, it follows that R is $*$-dense in A. Since R is regular, this suffices. ■

2. *ψA is E* in the "pushout" diagram below, and $R \equiv \rho A$, with $\rho \equiv \rho_A$, and $B \equiv \varepsilon R$ (the absolute of R, reviewed in 2.6), with $\varepsilon \equiv \varepsilon_R$. The maps $u_A \cdot \rho$ and $u_B \cdot \varepsilon$ are skeletal, and are coequalized by $q_s(x) \equiv x \vee s$, where

$$s = \vee \{ x \otimes y = u_A(x) \wedge u_B(y) \,:\, \rho_*(x) \wedge \varepsilon_*(y) = 0 \}.$$

Throughout, $E \equiv (\uparrow s)$, and we have the commutative diagram below, in which $w_A \equiv q_s \cdot u_A$ and $w_B \equiv q_s \cdot u_B$.

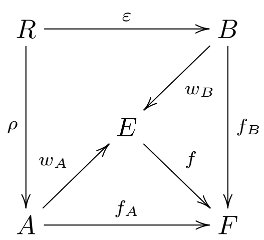
(2.4.1)



Establishing that (2.4.1) is a $\mathfrak{KNArS}$-pushout is the major achievement of Section 5 of [MZ08b]. Here we will indicate why this also works in $\mathfrak{KA}^2\mathfrak{S}$.

3. ([MZ08b, Lemma 5.3]) *Assume that A is a strongly joinfit frame. Suppose f_A and f_B are skeletal frame maps that make the outer square in the figure below commute.*

(2.4.2)



Then there is a unique frame homomorphism $f : E \longrightarrow F$ making the small triangles commute as well.

The proof is repeated verbatim, as it is the strong joinfitness that is used. Together with [MZ08b, Lemma 5.6], which shows that the induced f is skeletal, one shows that (2.4.1) is indeed the $\mathfrak{FrmS}$-pushout.

4. It then follows that w_A and w_B are dense, assuming A is strongly joinfit. (Refer to [MZ08b, Lemma 5.4], and observe what is lost in the generalization: the vertical maps in (2.4.1) need not be sections!)

5. ([MZ08b, Lemma 5.7]) *The map* $w_A : A \longrightarrow E$ *is epic in* $\mathfrak{K}\mathfrak{A}^2\mathfrak{S}$. The original proof goes through verbatim!

6. The preceding steps add up to a proof that ψ, defined by setting $\psi A = E$ and as reflection map $\psi_A = w_A$, is an epireflection of $\mathfrak{K}\mathfrak{A}^2\mathfrak{S}$ in the subcategory $\mathfrak{S}\mathfrak{P}\mathfrak{A}^2\mathfrak{S}$. Adding the argument of [M08b, Theorem 3.4], which invokes the Kuratowski-Mrówka Theorem for frames ([PT01, Chapter II, 6.5]), one shows that ψ_A is always a closed map, and therefore one-to-one. This uses compactness; normality does not play any role.

This completes the upgrade needed to prove the following theorem.

Theorem 2.5. *Let* A *be a compact strongly joinfit frame. Consider the diagram*

(2.5.1)
$$\begin{array}{ccc} \rho A & \xrightarrow{\varepsilon_{\rho A}} & \varepsilon(\rho A) \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle g_A} \\ A & \xrightarrow{\psi_A} & \psi A \end{array}$$

and assume that it forms a pushout in $\mathfrak{Frm}\mathfrak{S}$. *Then* ψ *defines a monoreflection of* $\mathfrak{K}\mathfrak{A}^2\mathfrak{S}$ *in the full subcategory* $\mathfrak{S}\mathfrak{P}\mathfrak{A}^2\mathfrak{S}$ *of all strongly projectable objects in* $\mathfrak{K}\mathfrak{A}^2\mathfrak{S}$. *More precisely, we have all of the following:*

(a) ψ_A *is a closed* $*$*-dense embedding.*

(b) $\varepsilon(\rho A) \cong \rho(\psi A)$, $g_A \cong \rho_{\psi A}$, *and* $\varepsilon_{\rho A} \cong \rho(\psi_A)$, *all three naturally; that is, the pushout in Figure (2.5.1) is equivalent to*

(2.5.2)
$$\begin{array}{ccc} \rho A & \xrightarrow{\rho(\psi_A)} & \rho(\psi A) \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \rho_{\psi A}} \\ A & \xrightarrow{\psi_A} & \psi A \end{array}$$

(c) ψA *is compact strongly joinfit and strongly projectable.*

(d) *If* A *is strongly projectable, then* ψ_A *is an isomorphism.*

(e) *If A is regular, then $\psi A \cong \varepsilon A$.*

(f) *If A is normal, then so is ψA.*

In particular, the restriction of ψ to regular frames is ε.

Remark 2.6. With regard to Theorem 2.5 we have the following supplementary comments:

1. Here ε stands for the absolute of a compact regular frame. In [MZ08a] it is established as the functorial epicompletion in $\mathfrak{KRegS}$, the category of compact regular frames with skeletal maps, with values in $\mathfrak{SPRegS}$, the full subcategory whose objects are strongly projectable.

2. The construction of ε may be found in [Ba88]. Recall that εA is $\mathrm{Idl}(\mathcal{P}A)$, the frame of ideals of the boolean frame $\mathcal{P}A$. The map ε_A is given by

$$\varepsilon_A(x) = \{\, a \in \mathcal{P}A \,:\, a \preceq x^{\perp\perp} \,\}.$$

In view of the remarks of 1.3(b), each ε_A is a closed embedding.

2.III. The Coreflection φ. Focussing on strongly joinfit frames, rather than normal joinfit ones, an interesting, but somewhat contrived phenomenon from [M08a] can be viewed in a much more natural light.

Remark 2.7. Suppose that A is any frame. Put

$$\varphi A \equiv \{\, x \in A \,:\, \rho_*(x)^\perp = x^\perp \,\}.$$

(In the following we leave off the subscript A on the adjoint ρ_*.) We refer the reader to [M08a, Theorem 2.11], where most of the claims below appear. The standing assumption there is that A is normal, but this hypothesis is not used in any of the proofs of the claims listed here.

1. $\varphi A = \{\, x \in A \,:\, \rho_*(x) \le x \le \rho_*(x)^{\perp\perp} \,\}$.

2. *φA is a strongly joinfit subframe of A.* Less is claimed in [M08a, Theorem 2.11], but the present affirmation is obvious.

3. $\rho(\varphi A) = \rho A$.

4. *Suppose that B is a frame and $h : A \longrightarrow B$ is a skeletal frame map. Then the restriction h' of h maps φA into φB. h' is the unique frame map making the square below commute.*

(2.7.1)
$$\begin{array}{ccc} \varphi A & \longrightarrow & A \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ \varphi B & \longrightarrow & B \end{array}$$

(The unlabeled horizontal maps are inclusions.)

5. *h' is skeletal.*

6. *If L is a strongly joinfit subframe of A, then $L \subseteq \varphi A$.* This both strengthens and simplifies a similar claim in [M08a, Theorem 2.11].

The foregoing facts lead to the following conclusion.

Theorem 2.8. *The functor φ defines a monocoreflection of $\mathfrak{Frm}\mathfrak{S}$ in the full subcategory $\mathfrak{A}^2\mathfrak{S}$ of strongly joinfit frames.*

2.IV. Other Upgrades. Briefly, we state the results that generalize [M08b, Theorems 4.1 & 4.5] to coherent, strongly joinfit frames, together with coherent, skeletal maps. The arguments of [M08b] carry over faithfully.

For the record, $\mathfrak{A}^2\mathfrak{Ch}\mathfrak{S}$ stands for the category of all coherent, strongly joinfit frames, together with coherent, skeletal maps. $\mathfrak{S}\mathfrak{P}\mathfrak{A}^2\mathfrak{Ch}\mathfrak{S}$ is the full subcategory whose objects are strongly projectable.

Theorem 2.9. *The functor ψ, restricted to the category $\mathfrak{A}^2\mathfrak{Ch}\mathfrak{S}$ and denoted $\psi^{\mathfrak{A}^2}$, defines a monoreflection in the subcategory $\mathfrak{S}\mathfrak{P}\mathfrak{A}^2\mathfrak{Ch}\mathfrak{S}$.*

And taking into account that each ψ_A is closed, one also has the following. Adding $\mathfrak{C}$ to the end of a categorical name denotes that the maps under consideration are closed.

Theorem 2.10. *The functor ψ, restricted to the category $\mathfrak{A}^2\mathfrak{Ch}\mathfrak{S}\mathfrak{C}$ and denoted $\psi^{\mathfrak{C}}$, defines a monoreflection in the subcategory $\mathfrak{S}\mathfrak{P}\mathfrak{A}^2\mathfrak{Ch}\mathfrak{S}\mathfrak{C}$.*

2.V. The Reflection d. Finally, in this section of upgrades, a brief one on the functor d, and of its relationship to epicompleteness. The references for the reader are to Sections 5 and 6 of [M08b]. All frames considered here are algebraic with the FIP. Define

$$d(x) = \bigvee \left\{ c^{\perp\perp} \,:\, c \le x,\ c \in \mathfrak{k}(L) \right\},$$

for each $x \in L$. The members of the quotient frame dL are called *d-elements.*

Remark 2.11. We now summarize the relevant features involving d. Most of this information may be found in [MZ03, 5.1] or [MZ06a, 4.2].

1. dL is an algebraic frame with the FIP; the compact elements of dL are the ones of the form $a^{\perp\perp} = d(a)$, with $a \in \mathfrak{k}(L)$. Infima in dL agree with those in L, whereas supremum is given by $\bigvee^d S \equiv d(\bigvee S)$, for any subset S of dL.

2. $d : L \longrightarrow dL$ is dense and coherent, and obviously $*$-dense.

3. On $\mathfrak{AlS}$, d is an epireflection in the subcategory $\mathfrak{AldS}$ of the $\mathfrak{AlS}$-objects for which $dL = L$. Restricted to $\mathfrak{ChS}$, d epireflects in the subcategory $\mathfrak{ChdS}$ consisting of all the coherent frames for which $dL = L$.

The frames for which $dL = L$ are the *d-frames.* It should be clear that L is a d-frame precisely when each compact element of L is a polar.

Next, we improve upon [M08b, Proposition 5.2].

Proposition 2.12. *Suppose that A is an algebraic frame with FIP. If A is strongly joinfit, then so is dA.*

Proof. We have the commutative square

(2.12.1)
$$\begin{array}{ccc} \rho A & \xrightarrow{\ \rho_A\ } & A \\ \big\downarrow{\scriptstyle \rho(d_A)} & & \big\downarrow{\scriptstyle d_A} \\ \rho dA & \xrightarrow{\ \rho_{dA}\ } & dA \end{array}$$

where the top and right arrows represent $*$-dense maps. It then follows easily that the bottom map is $*$-dense as well. ∎

We summarize the rest of the discussion about d from [M08b] in a remark.

Remark 2.13. Throughout this discussion the ambient category is $\mathfrak{A}^2\mathfrak{ChS}$.

(a) [M08b, Lemma 6.1] shows that the composite functor $e = d \cdot \psi$ is a reflection. From basic categorical principles one can then deduce that e epireflects in

$$\mathfrak{SPA}^2\mathfrak{ChS} \cap \mathfrak{A}^2\mathfrak{ChdS} = \mathfrak{SPRegS},$$

that is, in the subcategory of strongly projectable compact regular frames.

(b) The latter are epicomplete ([M08b, Proposition 6.3]). Thus,

$$\mathfrak{SPRegS} \subseteq E(\mathfrak{A}^2\mathfrak{ChS}) \subseteq \mathfrak{SPA}^2\mathfrak{ChS},$$

where $E(\mathfrak{A}^2\mathfrak{ChS})$ denotes the class of epicomplete objects.

(c) We know that $\mathfrak{SPRegS}$ is not monoreflective. We do not know whether $E(\mathfrak{A}^2\mathfrak{ChS})$ is reflective.

3 The Subfit Reflection.

We begin by reminding the reader of the so-called *saturation* nucleus s on a compact frame A: $s(a)$ is the supremum of all $x \in A$ such that

$$x \vee y = 1 \implies a \vee y = 1.$$

The compactness of A then insures that $s(a) \vee y = 1$ implies that $a \vee y = 1$.

It is well known that s is a nucleus. What we want is to regard s as a nuclear typing, in the sense of [MZ06a]; we shall explain.

In the commentary that follows we refer the reader to [Ba97] and [Ba02] for additional background. As in those articles, we assume frames are compact; on the other hand, we do not assume they are normal until later in this section.

Definition & Remarks 3.1. Denote the fixed set of s by SA, and the nucleus itself by s_A as a frame map onto SA.

1. As noted in [Ba02, 1.5], SA is *subfit*, in the sense that, for any $x < y$ there is a u such that $x \vee u < y \vee u = 1$. It is well known that a regular frame is subfit, and that the converse holds for normal frames.

2. The saturation nucleus captures joinfitness as follows: $s_A(x)$ is the meet of all $z \in A$, with $z \geq x$, such that $\uparrow z$ is joinfit. Refer to[M08a, §4] for further discussion; in particular, it is shown there how to define "saturation" in the noncompact case. Thus it is clear that A is joinfit precisely when s_A is dense.

3. Recall that a frame map f is *codense* if $f(x) = 1$ implies that $x = 1$. Observe that if $f : A \longrightarrow B$ is a codense frame map, then B is normal whenever A is normal. s_A is codense, and, in fact, up to isomorphism, it is the unique frame surjection onto a subfit frame.

4. Consider the frame homomorphisms $h : A \longrightarrow B$ between compact frames for which there is a frame homomorphism $s(h) : SA \longrightarrow SB$, obviously unique, making the square below commute:

(3.1.1)
$$\begin{array}{ccc} A & \xrightarrow{s_A} & SA \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle s(h)} \\ B & \xrightarrow{s_B} & SB \end{array}$$

It is easy to see that $s(h)$ exists if and only if $s_A(x) = s_A(y)$ in A implies that $s_B(h(x)) = s_B(h(y))$, and then, necessarily, $s(h)(s_A(x)) = s_B(h(x))$, for each $x \in A$. (These are the s-*natural* maps of [MZ06a].)

We do not have an illuminating characterization of s-natural maps, but here is an interesting class of them.

Lemma 3.2. *Let $h : A \longrightarrow B$ be a frame homomorphism of compact frames.*

(a) *If for each $x \in A$ and $u \in B$*

(3.2.1) $$h(x) \vee u = 1 \implies x \vee h_*(u) = 1,$$

then h is s-natural.

(b) *Each s_A satisfies (3.2.1).*

(c) *Each closed map satisfies (3.2.1).*

(d) *If h satisfies (3.2.1), then so does $s(h)$.*

Proof. (a) Suppose that the implication in (3.2.1) holds. If $s(x) = s(y)$ in A and $h(x) \vee u = 1$, then $x \vee h_*(u) = 1$, whence $y \vee h_*(u) = 1$. Apply h to the latter and conclude that $h(y) \vee u = 1$. Thus, $s(h(x)) \le s(h(y))$, and by symmetry the reverse also holds, proving that h is s-natural.

Since (b) is obvious, we move on to (c). If h is closed and $h(x) \vee u = 1$, then

$$1 = h_*(1) = h_*(h(x) \vee u) = x \vee h_*(u).$$

As for (d), if $s(h)(x) \vee u = 1$, with $x \in SA$ and $u \in SB$, then

$$1 = s_B(h(x)) \vee^{SB} u = s_B(h(x) \vee u),$$

and since saturation is codense, $h(x) \vee u = 1$, and so $x \vee h_*(u) = 1$. To conclude, observe that $h_* \cdot (s_B)_* = (s_A)_* \cdot s(h)_*$, and we have

$$x \vee s(h)_*(u) = x \vee (s_A)_*(s(h)_*(u)) = x \vee h_*((s_B)_*(u)) = x \vee h_*(u) = 1.$$

■

Remark 3.3. Call a frame homomorphism h *weakly closed* if (3.2.1) holds for h. The reader will doubtless have noted in the proof of the lemma that the implication in (3.2.1) is actually an equivalence.

As it is clear that the class of weakly closed maps is closed under composition, we may consider the category $\mathfrak{KFrmW}$ of compact frames with weakly closed maps.

In view of the comments in 3.1.2, whenever A is joinfit but not subfit, s_A is dense, but not closed; else s_A would be one-to-one.

We say that a property of frames $\mathcal{P}$ is *preserved by saturation* or, equivalently, *s-invariant* if SA satisfies $\mathcal{P}$ whenever A does. Next, we have a small catalogue of s-invariant properties. The fact that s is codense is what makes these work.

Proposition 3.4. *The following are preserved by saturation:*

(a) *Compactness; joinfitness; strong joinfitness;*

(b) *Normality together with joinfitness;*

(c) *Strong projectability together with joinfitness.*

Proof. Throughout A denotes a compact frame, and we exponentiate the join symbol by SA to indicate that this operation is being carried out there. We also drop the subscript on the nucleus s. Now, for any subset T of SA,

$$\bigvee{}^{SA} T = 1 \implies s\Big(\bigvee T\Big) = 1,$$

whence $\bigvee T = 1$.

It is now easy to see why compactness and joinfitness are s-invariant, and we carry out the proof for the latter. For assume A is compact and $0 < a \in SA$; pick $b < 1$ in A such that $a \vee b = 1$. One deduces that $s(b) < 1$ and $a \vee^{SA} s(b) = 1$. For the strong joinfitness we use the following commutative square:

(3.4.1)
$$\begin{array}{ccc} \rho A & \xrightarrow{\ \rho_A\ } & A \\ \Big\downarrow{\scriptstyle \rho(s)} & & \Big\downarrow{\scriptstyle s} \\ \rho SA & \xrightarrow{\ \rho_{SA}\ } & SA \end{array}$$

Since s_A and ρ_A are $*$-dense, it is clear that ρ_{SA} is as well. This establishes that SA is strongly joinfit, and with that (a).

Now suppose that A is both joinfit and normal; we show that SA is normal. To that end suppose $a \vee^{SA} b = 1$; as noted above, $a \vee b = 1$, so that there exist disjoint u

and v in A such that $a \vee v = u \vee b = 1$. Note that since A is joinfit, s is dense, whence $s(u) \wedge s(v) = 0$. Finally,

$$a \vee^{SA} s(v) = s(u) \vee^{SA} b = 1,$$

which proves (b).

For (c), suppose that A is strongly projectable and joinfit. Let x be a polar in SA; since saturation is dense, x is also a polar in A. Thus, x is complemented in A, and so also in SA. ■

With regard to the next result, let us hasten to add that s is made functorial on compact normal frames in [Ba02] as well, but the action – on morphisms! – is different from the present one.

Proposition 3.5. *s is an epireflection of $\mathfrak{KArW}$ in $\mathfrak{SFitW}$, the subcategory whose objects are subfit.*

Proof. The uniqueness of the induced $s(h)$ in the defining diagram insures that s is functorial with weakly closed maps. The fact that s is a closure operator means that it is epireflective.

Since all frames are joinfit, each $s_A : A \longrightarrow SA$ is a frame homomorphism. Finally, Lemma 3.2(b) tells us that each s_A is in $\mathfrak{KArW}$. ■

We have a good deal of what we need for the main result of this section. Since the theorem composes ψ with s, some caution must be observed with the maps; that is, they should be skeletal as well as weakly closed. Thus, the categories under consideration all have names ending in $\mathfrak{WS}$. The reader would do well to review (in 1.4) the conventions on the naming of categories.

One last preliminary is required, which the reader surely has anticipated: checking that s preserves skeletal maps and ψ preserves weakly closed maps. The first is easy; for the second, we adapt Lemmas 4.3 and 4.4 of [M08b] to weakly closed maps. Thus we have a sequence of lemmas, the culmination of which is Lemma 3.8.

Suppose that $h_i : A_i \longrightarrow B_i$ $(i = 1, 2)$ are frame maps; the induced map $h_1 \coprod h_2$ is the one such that $(h_1 \coprod h_2)(a_1 \otimes a_2) = h_1(a_1) \otimes h_2(a_2)$.

The proof of the first lemma is a combinatorial exercise involving coproducts, which we shall leave to the reader. Recall that a frame is *zero-dimensional* if every element is a join of complemented elements. Each strongly projectable compact regular frame is, evidently, zero-dimensional.

Lemma 3.6. *Suppose that $h : A \longrightarrow B$ is a weakly closed frame homomorphism between compact strongly joinfit frames. Then, for each compact zero-dimensional frame E, the map $h \coprod 1_E : A \coprod E \longrightarrow B \coprod E$ is weakly closed.*

Next, we generalize [M08b, Lemma 4.4].

Lemma 3.7. *Suppose that $h \cdot g$ is a weakly closed composite of frame maps, with g surjective. Then h is weakly closed.*

Proof. Suppose that $h(b) \vee y = 1$; let $b = g(g_*(b))$, so that $h(g(g_*(b))) \vee y = 1$, and hence $g_*(b) \vee g_*(h_*(y)) = 1$, since $h \cdot g$ is weakly closed. Applying g to the preceding identity, we get that $b \vee h_*(y) = 1$. ■

Lemma 3.8. *Suppose that $h : A \longrightarrow B$ is a frame homomorphism between compact strongly joinfit frames.*

(a) *If h is skeletal then so is $s(h)$.*

(b) *If h is weakly closed then so is $\psi(h)$.*

Proof. (a) Suppose that $x \in SA$ and $x^{\perp\perp} = 1$ in SA; then, also, $x^{\perp\perp} = 1$ in A, because s is dense. Thus,

$$s(h)(x)^{\perp\perp} = s_B(h(x))^{\perp\perp} = 1,$$

proving that $s(h)$ is skeletal.

(b) To begin, $\varepsilon(\rho(h))$ is closed, being a frame map between compact regular frames ([M08b, Lemma 4.2]). Then

$$h \coprod \varepsilon(\rho(h)) = \Big(h \coprod 1_{\varepsilon(\rho B)}\Big) \cdot \Big(1_A \coprod \varepsilon(\rho(h))\Big),$$

which is weakly closed, as it is a product of a weakly closed map (Lemma 3.6) and a closed one ([M08b, Lemma 4.3]).

From here one proceeds almost verbatim as in the proof of [M08b, Theorem 4.5] to show that $\psi(h)$ is weakly closed. ■

Theorem 3.9. *On $\mathfrak{KNArWS}$, the reflections s and ψ commute. The composite $\psi \cdot s$ epireflects in*

$$\mathfrak{SPArWS} \bigcap \mathfrak{KRegS} = \mathfrak{SPRegS}.$$

Proof. As we have already noted in 2.13(a), whenever two reflections α and β commute, then $\alpha \cdot \beta$ is a reflection in the intersection of the reflected subcategories in question. To prove that $\alpha \cdot \beta = \beta \cdot \alpha$ it suffices to show that α leaves invariant the subcategory that β reflects in and *vice versa.*

Suppose A is a compact normal, joinfit frame. Proposition 3.4 guarantees that SA also has these properties, and since SA is subfit, it is, in fact, regular. Thus s reflects $\mathfrak{KNArWS}$ in $\mathfrak{KRegS}$, and $\psi SA = \varepsilon SA$ is again regular, proving that ψ leaves $\mathfrak{KRegS}$ invariant. Proposition 3.4(c) and Lemma 3.8 together complete the proof that s and ψ commute. ■

We conclude this section with some remarks about the scope of the foregoing. First, we show that $\mathfrak{SPRegS}$ is not monoreflective in $\mathfrak{KNArWS}$; thus, the present situation rather resembles that of 2.13 involving d and ψ. And to demonstrate that $\mathfrak{SPRegS}$ is not monoreflective, it is enough to show that s is not a monoreflection. Since it is not known whether monomorphisms are one-to-one in $\mathfrak{KNArWS}$, this seems like a substantial achievement.

Remark 3.10. In [Ba97, Lemma 1.4] it is shown that, for every compact normal frame A, s_A is a retraction. More precisely, the function defined by

$$r_A(x) = \vee \{ a \in A : a \preceq x \}$$

is a frame homomorphism, and $s_A \cdot r_A = 1$.

There is more to r_A than meets the eye; we collect some observations as follows.

Proposition 3.11. *Suppose that A is a compact normal frame.*

(a) $s_A = (r_A)_*$.

(b) *Further, if A is also joinfit, then r_A is closed and $*$-dense.*

Proof. (a) To begin, observe the following; first, let us drop the subscript A on both r and s. We show that $r(s(x)) \leq x$, for each $x \in A$; by definition, $r(s(x)) = \vee \{ a \in A : a \preceq s(x) \}$, and

$$a \preceq s(x) \implies a \preceq x,$$

as there exists a $b \in A$ such that $a \wedge b = 0$ and $s(x) \vee b = 1$, whence $x \vee b = 1$ and the claim. Thus,

$$r(s(x)) = \vee \{ a \in A : a \preceq x \} \leq x.$$

Now, $y \leq s(x)$ implies that $r(y) \leq r(s(x)) \leq x$, and the reverse implication holds because $s \cdot r = 1$. It follows that $s = r_*$.

(b) Suppose now that A is also joinfit. First, since $s(0) = 0$, s is a frame homomorphism $A \longrightarrow SA$, and any frame embedding whose right adjoint is a frame homomorphism is necessarily closed. Finally, since s is dense, r is $*$-dense. ∎

We now have the desired consequence.

Corollary 3.12. $\mathfrak{SPRegS}$ *is not monoreflective in* $\mathfrak{KNArWS}$.

Proof. The upshot of the preceding proposition is that each s_A is a retraction in $\mathfrak{KNArWS}$. Any monic retraction is an isomorphism. ∎

The reader may legitimately wonder why this recent development abandons the strongly joinfit frames for the more particular normal joinfit ones. Here are some of the reasons.

Remark 3.13. On extending beyond compact normal, joinfit frames in Theorem 3.9 and the subsequent remarks, it should be noted that

1. Theorem 3.9 depends heavily on the fact that normal subfit frames are regular; if the domain of the functors is extended to strongly joinfit frames, $\psi \cdot s$ reflects in the subcategory of strongly joinfit, strongly projectable, subfit frames. Such frames are not necessarily regular; see Example 4.7.

2. The proof in [Ba97] that the map r_A introduced above preserves finite joins relies on the hypothesis of normality, as does the retractive identity $s_A \cdot r_A = 1$. Nonetheless, all that Corollary 3.12 requires for s to fail to be monorefective, is one object that is normal and joinfit, but not regular.

4 Applications: the Frame of Radical Ideals.

The objective in this section is to illustrate how the scope of the discussion is broadened by generalizing from normal joinfit frames to those that are strongly joinfit. This is done in the context of coherent frames, and, concretely, through the frame of radical ideals of a commutative ring with identity. By a theorem of Hochster ([Ho69]), every coherent frame arises as such a frame. Banaschewski, in [Ba96], gives a *Choice-free* proof of this result.

Definition & Remarks 4.1. All rings are commutative and possess a multiplicative identity 1. Let R denote such a ring, and $\mathrm{Rad}(R)$ the frame of all radical ideals.

Several references for this discussion may be found in [M08a, §6].

1. An ideal $\mathfrak{a}$ of R is *radical* if $x^2 \in \mathfrak{a}$ implies $x \in \mathfrak{a}$. It is easy to see that the ideal $\mathfrak{a}$ is radical precisely when $R/\mathfrak{a}$ is *semiprime*, that is, there are no nonzero nilpotent elements. With *Choice*, $\mathfrak{a}$ is radical if and only if it is an intersection of prime ideals.

   It is well known that $\mathrm{Rad}(R)$ is a coherent frame.

2. Assuming *Choice*, $\mathrm{Rad}(R)$ is joinfit if and only if R is *semisimple*, that is, the intersection of the maximal ideals of R is trivial.

3. $\mathrm{Rad}(R)$ is normal precisely when R is *Gelfand*: if $a + b = 1$ in R, then there exist $r, s \in R$ such that
$$(1 - ra)(1 - sb) = 0.$$
   Assuming *Choice*, R is Gelfand if and only if every prime ideal is contained in a unique maximal ideal.

4. With *Choice*, Rad(R) is subfit if and only if every prime ideal is an intersection of maximal ideals.

Let $\mathbb{Z}$ denote the ring of integers. Observe that since $\mathbb{Z}$ is a semisimple principal ideal domain, Rad($\mathbb{Z}$) is subfit, yet its regular coreflection is trivial. Thus, Rad($\mathbb{Z}$) is far from being strongly joinfit.

The example that follows is due to Warren McGovern.

Example 4.2. *A coherent strongly joinfit frame that is not normal.*

Once again R stands for a commutative ring with 1. Rad(R) is strongly joinfit as long as every nonzero ideal contains a nonzero idempotent. Note that an idempotent generates a complemented ideal. Furthermore, observe that any complemented element of a frame lies in the regular coreflection.

Now let R be the ring of all real sequences which are eventually integers and constant. Thus, $f \in R$ if and only if there exist integers $k > 0$ and m such that $f(n) = m$, for every $n \geq k$. It should be clear that R is semisimple and that every principal ideal contains a nonzero idempotent.

On the other hand, R is not Gelfand. To see this, consider the constant sequences 3 and -2. Let $f, g \in R$ be eventually k and m, respectively. Then $(1 - 3f)(1 + 2g)$ is eventually $(1 - 3k)(1 + 2m)$, which is not zero!

To conclude this section, we have an example (4.7), as promised in 3.13. It will be useful to record the behavior under extension as well as under formation of subframes, of several of the properties discussed in this paper. We begin with some features of $*$-dense extensions. The proof of the following lemma may safely be left as an exercise.

Lemma 4.3.

(a) *If $m : A \longrightarrow B$ and $n : B \longrightarrow C$ are frame extensions, then $n \cdot m$ is $*$-dense if and only m and n are $*$-dense.*

(b) *For any frame A, the union of any updirected set of $*$-dense extensions is $*$-dense.*

Proposition 4.4. *Suppose that $m : A \longrightarrow B$ is a $*$-dense frame embedding.*

(a) *If A is joinfit, then B is also joinfit.*

(b) *If A is strongly joinfit, then so is B.*

Proof. (a) Begin with $b > 0$ in B. Then since m is $*$-dense and A is joinfit, $m_*(b) > 0$ and there is an $x < 1$ in A, such that $m_*(b) \vee x = 1$; applying m,

$$1 = m(m_*(b)) \vee m(x) \leq b \vee m(x),$$

so that $b \vee m(x) = 1$, proving (a).

For (b), we have the commutative square

$$(4.4.1)\qquad \begin{array}{ccc} \rho A & \xrightarrow{\rho_A} & A \\ \Big\downarrow{\scriptstyle \rho(m)} & & \Big\downarrow{\scriptstyle m} \\ \rho B & \xrightarrow{\rho_B} & B \end{array}$$

with the top and right arrows being $*$-dense. An application of Lemma 4.3(a) then insures the other two are $*$-dense as well. ■

Going the other way, we have similar behavior for weakly closed embeddings. As we demonstrate, the claims of Proposition 4.5 are true for dense weakly closed maps – such as the saturation s, for (a), (b), and (c).

We have no examples showing that (a) in the proposition that follows fails for strong joinfitness. We doubt that (d) is true for embeddings without the hypotheses of coherence, but we have found no counterexamples either. Note that s shows that (d) fails in general without coherence.

Proposition 4.5. *Suppose that $m : A \longrightarrow B$ is a weakly closed frame embedding.*

(a) *If B is joinfit then so is A.*

(b) *If B is subfit then so is A.*

(c) *If B is normal then so is A.*

(d) *If m is also a coherent map between algebraic frames, and B is regular, then so is A.*

Proof. The proof of (a) is very similar to those of (b) and (c), as well as the easiest of the three. We leave it as an exercise.

(b) Suppose $a < b$ in A. Owing to the subfitness of B, there is a $y \in B$ such that $m(a) \vee y < 1$, while $m(b) \vee y = 1$. Then $a \vee m_*(y) < 1$, and since m is weakly closed, $b \vee m_*(y) = 1$.

(c) Assume $a \vee b = 1$ in A. There are disjoint $u, v \in B$ such that $m(a) \vee v = 1 = u \vee m(b)$. Since m is weakly closed, we have $a \vee m_*(v) = 1 = m_*(u) \vee b = 1$, and $m_*(u) \wedge m_*(v) = m_*(u \wedge v) = m_*(0) = 0$, proving that A is normal.

(d) An algebraic frame is regular precisely when every compact element is complemented ([MZ03, Theorem 2.4(a)]). So, if $a \in \mathfrak{k}(A)$, there is a $y \in B$ such that $m(a) \wedge y = 0$ and $m(a) \vee y = 1$. Since $a \leq m_*(m(a))$, it follows that $a \vee m_*(y) = 1$ and $a \wedge m_*(y) = 0$. ■

Remark 4.6. It should be observed that (c) and (d) in the preceding result amount to converses for (f) and (e), respectively, in Theorem 2.5.

Example 4.7. *A compact subfit frame which is strongly projectable, strongly joinfit, but not regular.*

Let $B = S\psi\mathrm{Rad}(R)$, where R is the ring of Example 4.2.

Let us first consider $A \equiv \psi\mathrm{Rad}(R)$: as $\mathrm{Rad}(R)$ is strongly joinfit and $\psi \equiv \psi_{\mathrm{Rad}(R)}$ is a closed $*$-dense embedding, A is also strongly joinfit, by Proposition 4.4(b). A is not normal, because of Proposition 4.5(c). Note that A is strongly projectable.

Next, apply saturation and Proposition 3.4 to conclude that B is subfit, strongly joinfit and strongly projectable. However, as the map $s_A : A \longrightarrow B$ is dense and weakly closed (Lemma 3.2(b)), the proof of Proposition 4.5(c) shows that B is not normal, and hence not regular either.

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Department of Mathematics, University of Florida, Box 118105
Gainesville, FL 32611-8105
email: martinbad@math.ufl.edu