Patch-Generated Frames and Projectable Hulls.[1]

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For Bernhard Banaschewski, on the occasion of his 80th birthday.

Abstract. This article considers coherent frame homomorphisms $h : L \longrightarrow M$ between coherent frames, which induce an isomorphism between the boolen frames of polars, with M projectable, and such that M is generated by $h(L)$ and certain complemented elements of M. This abstracts the passage from a semiprime commutative ring with identity to its projectable hull. The frame theoretic setting is investigated thoroughly, first without any assumptions beyond the Zermelo-Fraenkel axioms of set theory, and, subsequently, assuming that algebraic frames are spatial. The culmination of this effort is the result that the spectrum of d-elements of M is obtained from that of L by refining the given hull-kernel topology to the patch topology.

The second part of the article relates the projectable hull to the (von Neumann) regular hull, in a variety of contexts, including that of f-rings. For a uniformly complete f-algebra A, it is shown that the maximal ℓ-ideals of A that are traces of real maximal ideals of the regular hull HA are precisely the almost P-points of the space of maximal ℓ-ideals of A.

This research was motivated by our desire to understand the least von Neumann regular extension of a semiprime commutative ring with identity. One realizes fairly quickly, however, that the smaller projectable hull is rather narrowly associated with it, and the focus narrows upon the projectable hull, and on the relationship between its spectrum and that of the original ring. There is a more intimate connection between the two structures, however, which is best rendered in a frame-theoretic setting. This is the subject matter for a good part of the paper: a detailed analysis of the passage from a coherent frame L, by means of a dense and $*$-dense frame homomorphism, to a projectable frame M.

The paper consists of two parts: the first, as already described, is purely frame-theoretic (over Sections 1, 2, and 3) – at first purely lattice-theoretic, prior to consideration of the prime spectra, under the assumption of the Axiom of Choice. The second part considers the applications: to semiprime commutative rings with identity,

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in a number of more restrictive settings, such as for f-rings, culminating with the representation of archimedean f-rings, and how it is extended to the projectable hull and the regular hull.

We defer an introduction to the projectable and regular hulls until the frame-theoretic matters have been dealt with, and straightaway proceed to the frame-theoretic elements of the investigation.

1 Frame-theoretic Resources.

A great deal of what makes the structures of the regular and projective hulls what they are is frame-theoretic, and here we develop that background material. The context is that of algebraic frames with the finite intersection property (abbr. FIP). The particular references that will play a role in this article are [MZ03, M06, MZ06b].

Definition 1.1. We begin with a short frame-theoretic dictionary. For additional references, the reader should appeal to [J82].

Throughout, L is a frame, in which the top and bottom are denoted 1 and 0, respectively. For $x \in L$, denote the set of elements of L less than or equal to (resp. greater than or equal to) x by $\downarrow x$ (resp. $\uparrow x$).

- L is *algebraic*: each member is a supremum compact elements. Throughout, $\mathfrak{k}(L)$ stands for the set of compact elements in the lattice L.
- L has the *finite intersection property* (abbr. *FIP*): for any pair $a, b \in \mathfrak{k}(L)$ it follows that $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema.
- L is *coherent*: 1 is compact and L has the FIP.
- $p \in L$ is *prime*: $p < 1$ and $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. Spec(L) shall denote the set of prime elements of L.
- The operation $a \to b$ (in a frame L):

  $$a \to b = \vee \{ x \in L \,:\, a \wedge x \leq b \}.$$

  Also, $a^\perp \equiv a \to 0$.
- $p \in L$ is a *polar*: it is of the form $p = y^\perp$, for some $y \in L$.

  It is well known that the set PL of all polars forms a complete boolean algebra, in which infima agree with those in L.
- $a \preceq b$: (in a frame) $b \vee a^\perp = 1$.

  $x \in L$ is *regular*: $x = \vee \{ a \in L \,:\, a \preceq x \}$.

  A frame L *regular*: each element of L is regular.

- In a frame L: a closure operator j is a *nucleus* if $j(a \wedge b) = j(a) \wedge j(b)$. It is well known that j is a nucleus $\iff$ in the fixed set jL: $x \in jL$ implies that $a \to x \in jL$, for each $a \in L$.

- An algebraic frame L has *disjointification*: for each pair of compact elements $a, b \in L$, there exist $c \wedge d = 0$ in $\mathfrak{k}(L)$, such that $c \leq a$ and $d \leq b$, and $a \vee b = a \vee d = c \vee b$.

  If L has disjointification then, for each prime $p \in L$, $\uparrow p$ is a chain, and the converse is true as long as L has the FIP ([Mo54]). When $\uparrow p$ is a chain for each $p \in \mathrm{Spec}(L)$, we say that $\mathrm{Spec}(L)$ is a *root system*.

Definition & Remarks 1.2. The category of discourse is $\mathfrak{AFrm}$, consisting all algebraic frames with the FIP, and the frame homomorphisms that take compact elements to compact elements, which we will call *coherent*.

If $h : L \longrightarrow M$ is any frame homomorphism, then $h_* : M \longrightarrow L$ denotes its adjoint; that is, the map defined by

$$x \leq h_*(y) \iff h(x) \leq y, \quad \text{for all} \quad x \in L, \ y \in M.$$

The above defines h_* unambiguously because h preserves suprema and L is join-complete.

The following are well known:

1. h_* preserves all infima.

2. $j \equiv h_* \cdot h$ is a nucleus; the fixed set of j, jL, is a frame isomorphic to the image $h(L)$, and $h|_{jL}$ witnesses this, with inverse $h_*|_{h(L)}$.

3. h is one-to-one if and only if $h_* \cdot h = 1_L$, and that h is surjective if and only if $h \cdot h_* = 1_M$.

4. h is *dense* if $h(x) = 0$ implies $x = 0$. Then the following are equivalent: (i) h is dense; (ii) j is dense; (iii) $h_*(0) = 0$.

5. If h is a $\mathfrak{AFrm}$-map, then jL is an algebraic frame with the FIP.

We also say that h is $*$-*dense* if $h_*(y) = 0$ implies that $y = 0$. Note that this follows if h is surjective, because h_* is one-to-one.

Next, we introduce a subcategory of $\mathfrak{AFrm}$ which will play an important role in the following.

Definition & Remarks 1.3. (a) Let $h : L \longrightarrow M$ be a frame homomorphism. Call h *skeletal* if whenever $x \in L$ is dense, then $h(x)$ is dense in M. ($x \in L$ is *dense* if $x^\perp = 0$.) The term "skeletal" is motivated by the corresponding concept in topology.

It is well known – see [BaP96] – that h is skeletal if and only if

$$x_1^{\perp} = x_2^{\perp} \;\Rightarrow\; h(x_1)^{\perp} = h(x_2)^{\perp}.$$

Alternatively, setting $p_L(x) = x^{\perp\perp}$, for $x \in L$, we have that h is skeletal precisely when there exists a frame homomorphism $P(h) : PL \longrightarrow PM$ such that the square below commutes:

(1.3.1)
$$\begin{array}{ccc} L & \xrightarrow{\;h\;} & M \\ \Big\downarrow{\scriptstyle p_L} & & \Big\downarrow{\scriptstyle p_M} \\ PL & \xrightarrow{\;P(h)\;} & PM \end{array}$$

The induced $P(h)$ is, evidently, unique, and is a morphism of boolean frames. In fact, P defines a reflection of the category $\mathfrak{Frm}_s$ of frames with skeletal maps in the full subcategory of boolean frames ([BaP96]).

(b) We define a nucleus, derived from p, in the context of $\mathfrak{AFrm}$. Put

$$d(x) = \bigvee \{\, a^{\perp\perp} \;:\; a \leq x,\; a \in \mathfrak{k}(L) \,\}.$$

It is well established – see [MZ03, §5] – that d is a nucleus and that the fixed set, dL, is an algebraic frame with FIP, in which the supremum operation is defined by

$$\bigvee^{dL} S \equiv d\Big(\bigvee^{L} S\Big).$$

Thus, $d : L \longrightarrow dL$ is a coherent frame map, and $\mathfrak{k}(dL) = p_L(\mathfrak{k}(L))$.

The members of dL are the *d-elements*. The nucleus d – indeed, the choice of the letter 'd' – was motivated by work of Huijsmans and de Pagter on Riesz spaces in [HdP80a, HdP80b].

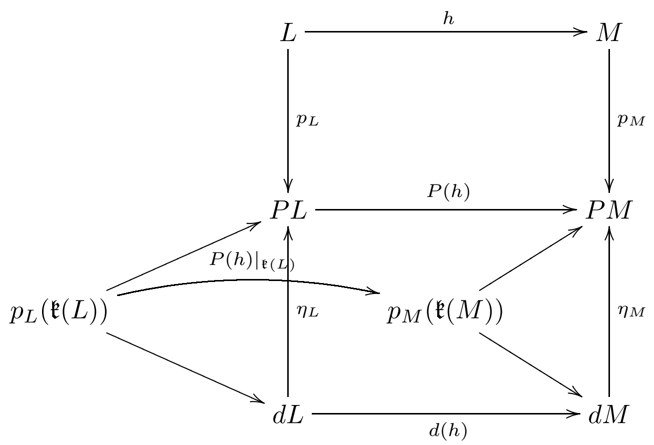
If the $\mathfrak{AFrm}$-morphism $h : L \longrightarrow M$ is skeletal, then it also induces an $\mathfrak{AFrm}$-morphism by

$$d(h)(x) = \bigvee \{\, h(a)^{\perp\perp} \;:\; a \leq x,\; a \in \mathfrak{k}(L) \,\}.$$

The following commutative diagram may be helpful in tracing the "provenance" of

$d(h)$:

(1.3.2)



(In the diagram the unlabelled arrows are inclusions; η_L is the induced nucleus $\eta_L(x) = p_L(x)$, for $x \in dL$.)

In conclusion of these remarks, we mention the work in [MZ06a], where the foregoing material is discussed in greater generality.

The following simple lemma is very useful throughout.

Lemma 1.4. *Suppose that $h : L \longrightarrow M$ is a skeletal $\mathfrak{AFrm}$-map between algebraic frames with FIP, such that $P(h) : PL \longrightarrow PM$ is one-to-one. Then $d(h) : dL \longrightarrow dM$ is one-to-one.*

Proof. Suppose $d(h)(x) = d(h)(y)$, with $x, y \in dL$. Then, for each compact $a \le x$, there is a compact $b \le y$ such that $P(h)(p_L(a)) \le P(h)(p_L(b))$; this uses compactness, along with the fact that the supremum defining d in 1.3(b) is up-directed. Then, per our assumption, $p_L(a) \le p_L(b)$, which implies that $x \le y$; by symmetry, $y \le x$, and we are done. ∎

Remark 1.5. It is obvious that if $h : L \longrightarrow M$ is surjective and skeletal, then $P(h)$ too is surjective. Further, applying [MZ06a, Proposition 4.3], it follows that $d(h)$ is surjective if and only if the restriction of $P(h)$ to $p_L(\mathfrak{k}(L))$ is onto $p_M(\mathfrak{k}(M))$.

We shall investigate what happens, for a given skeletal $\mathfrak{AFrm}$-map $h : L \longrightarrow M$, in the event that $P(h)$ is an isomorphism; Lemma 1.4 guarantees that $d(h)$ is one-to-one, but as we shall see, it need not be surjective.

2 p-Isomorphisms.

Definition 2.1. Suppose that $h : L \longrightarrow M$ is a skeletal frame homomorphism. We say h is a *p-isomorphism* if $P(h)$ is an isomorphism.

A basic lemma to start with.

Lemma 2.2. *Suppose that $h : L \longrightarrow M$ is a frame homomorphism which is both dense and $*$-dense. Then h satisfies*

$$h(x) \wedge y = 0 \quad \Longleftrightarrow \quad x \wedge h_*(y) = 0,$$

for all $x \in L$ and all $y \in M$. In particular, h is skeletal.

Proof. Let $x \in L$ and $y \in M$. If $h(x) \wedge y = 0$, then applying h_* and h, and the density of h, we get

$$0 = h(h_*(h(x) \wedge y)) = h(x) \wedge h(h_*(y)) = h(x \wedge h_*(y)),$$

whence $x \wedge h_*(y) = 0$. For the converse use the $*$-density of h.

To conclude that h is skeletal, it suffices to show that if $x \in L$ is dense then so is $h(x) \in M$ (1.3(a)). But this is obvious from the equivalence just established ■

The next result is pivotal for our understanding of the projectable and regular hulls of commutative rings.

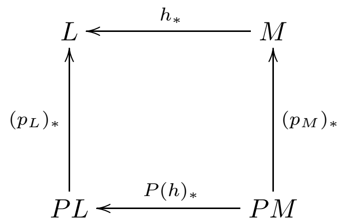
Theorem 2.3. *Suppose that $h : L \longrightarrow M$ is a frame homomorphism which is dense as well as $*$-dense. Then h is a p-isomorphism.*

Proof. The preceding lemma insures that h is skeletal. Now, the map $P(h) : PL \longrightarrow PM$ is a morphism of boolean frames. Observe first that it preserves complements: by the remarks in 1.3(a), and since $x \vee x^\perp$ is dense, for each polar $x \in L$, $h(x) \vee h(x^\perp)$ is dense in M. This suffices to show that $P(h)(x^\perp) = h(x)^\perp$. Thus, $P(h)$ is a boolean morphism.

Then, to show that $P(h)$ is one-to-one it suffices to prove it is dense: if $P(h)(x) = 0$, with $x \in PL$, then $h(x)^{\perp\perp} = 0$, which obviously implies $x = 0$, using the density of h. Thus, we also have that $P(h)_* \cdot P(h) = 1$ (1.2), which suggests that $P(h)_*$ ought to be the inverse of $P(h)$.

Now $P(h)_*$ is surjective, and preserves infima. What is required then, is a proof that it too is a boolean map which is dense. We begin by describing $P(h)_*$; the reader is referred to diagram (1.3.1) in 1.3(a). The diagram that follows is obtained by taking adjoints.

(2.3.1)



Note that the adjoints of the nuclei p are inclusions, making $P(h)_*$ the restriction of h_* to PM. Thus, assuming for a moment that $P(h)_*$ is a boolean map, it should be clear that the $*$-density of h implies that of $P(h)$.

Finally, we establish that $h_*(y^\perp) = h_*(y)^\perp$, which shows that $P(h)_*$ preserves complements, and completes the proof of the theorem. First, since h_* preserves meets, it is clear that $h_*(y^\perp) \leq h_*(y)^\perp$. As to the reverse inequality, suppose $a \in L$, such that $a \wedge h_*(y) = 0$. Then, by Lemma 2.2, $h(a) \wedge y = 0$, that is,

$$a \leq h_*(h(a)) \leq h_*(y^\perp),$$

proving that $h_*(y)^\perp \leq h_*(y^\perp)$. ∎

Remark 2.4. A remark upon the proof and the scope of Theorem 2.3 is in order. First, the proof of the theorem shows that if h is dense, then $P(h)$ is a one-to-one boolean map. Conversely, if h is skeletal and $P(h)$ is a boolean embedding, it is easily seen that h is dense.

The $*$-density of h seems responsible for making the adjoint $P(h)_*$ one-to-one. On the other hand, we have not been able to show that the $*$-density of h is also necessary for $P(h)_*$ to be one-to-one.

Next, we have a corollary to Theorem 2.3 and a proposition. The corollary is an immediate consequence of the theorem and Lemma 1.4.

Corollary 2.5. *Suppose that $h : L \longrightarrow M$ is an $\mathfrak{AFrm}$-map which is both dense and $*$-dense. Then h induces the coherent frame embedding $d(h) : dL \longrightarrow dM$.*

With regard to the operator d, one has the notion of a "d-isomorphism", a stronger condition than the combination of density and $*$-density. That is the subject of Proposition 2.6. We define: the $\mathfrak{AFrm}$-map $h : L \longrightarrow M$ is a *d-isomorphism* if $d(h)$ is an isomorphism.

Proposition 2.6. *Suppose that $h : L \longrightarrow M$ is a dense $\mathfrak{AFrm}$-map. Then h is a d-isomorphism if and only if*

(2.6.1) $$\forall\, b \in \mathfrak{k}(M),\ \exists\, a \in \mathfrak{k}(L),\ b^{\perp\perp} = h(a)^{\perp\perp}.$$

Proof. If $d(h) : dL \longrightarrow dM$ is an isomorphism, it restricts to an isomorphism of their sublattices of compact elements; that is, from $p(\mathfrak{k}(L))$ onto $p(\mathfrak{k}(M))$. This clearly shows that (2.6.1) is necessary.

Conversely, if the condition holds, then it is an easy exercise to show that h is skeletal (using the density of h). By the comments in 2.4, $P(h)$ is one-to-one, and Lemma 1.4 then insures that $d(h)$ is also one-to-one. But (2.6.1) implies $d(h)$ is surjective, proving h is d-esssential. ∎

Remark 2.7. Condition (2.6.1) is known in the literature as "rigidity" ([CM90]), and is associated with the embedding of one ordered algebraic structure in another. It is stronger than "p-isomorphy". We return to this notion in 3.6 and 7.8.

We wish to consider p-isomorphisms $h : L \longrightarrow M$ in the category $\mathfrak{CohFrm}$, of all coherent frames and coherent frame maps, into projectable frames M. We have in mind rings of quotients and maps of this kind between their frames of radical ideals.

The question of the existence of p-isomorphisms to projectable frames will be settled in the next section. We define "projectable" frame next; for amplification the reader is referred to [MZ03, MZ06b].

Definition & Remarks 2.8. Let L be an algebraic frame. L is *projectable* if for each compact $c \in L$, $1 = c^{\perp\perp} \vee c^{\perp}$. It follows from [MZ03, Theorem 2.4] that L is projectable if and only if every d-element is regular, and [MZ03, 5.4] tells us that if L is projectable, then dL is a regular frame, as well as a subframe of L, namely, the subframe of L generated by the set of complemented elements $\{ c^{\perp\perp} : c \in \mathfrak{k}(L) \}$ ([MZ06b, 8.4]).

The frame-theoretic model for the projectable hull of a commutative ring with identity is, in fact, a $\mathfrak{CohFrm}$-map $h : L \longrightarrow M$ which

- is both dense and $*$-dense,
- such that M is projectable, and
- M is the subframe generated by $h(L)$ and $E(L)$, where $E(L)$ is the boolean algebra generated by $\{ h(c)^{\perp\perp} : c \in \mathfrak{k}(L) \}$.

The reader should keep in mind that, for each $c \in \mathfrak{k}(L)$, $h(c)$ is compact, and so $h(c)^{\perp\perp}$ is complemented in M.

For brevity, as well as other reasons, we shall say, in the collective presence of the properties described in the preceding paragraph, that M is *patch-generated* by h, or, alternatively, that h *patch-generates* M.

We conclude this section with a result that summarizes the arithmetical features of a projectable M which is patch-generated by a frame map which is both dense and $*$-dense.

Proposition 2.9. *Suppose that $h : L \longrightarrow M$ is a $\mathfrak{CohFrm}$-morphism which is both dense and $*$-dense, and maps to the projectable frame M. Then h patch-generates M if and only if each $x \in M$ may be expressed*

$$x = \bigvee \Big\{ h(a_\lambda) \wedge e_\lambda \,:\, a_\lambda \in \mathfrak{k}(L),\ e_\lambda \in E(L) \Big\}.$$

If h patch-generates M, then, in addition, we have the following:

(a) *Each compact element of M is of the form*

$$(h(a_1) \wedge e_1) \vee \cdots \vee (h(a_n) \wedge e_n),$$

where each $a_i \in \mathfrak{k}(L)$ and each $e_i \in E(L)$, and, furthermore, the set $\{ e_1, \ldots, e_n \}$ is pairwise disjoint.

(b) *h induces the $\mathfrak{CohFrm}$-embedding $d(h) : dL \longrightarrow dM$, with dM regular, such that each $d(h)(c)$ $(c \in \mathfrak{k}(dL))$ is complemented, and $\mathfrak{k}(dM) = E(L)$.*

Proof. First, note that h preserves finite meets and maps compact elements to compact elements; L has the FIP; $h(L)$ is a coherent subframe of M; and $E(L)$ is a sublattice of M. These facts, together with the frame law give us the first assertion.

As to (a), observe that $E(L) \subseteq \mathfrak{k}(M)$, which makes it clear that any expression

$$(h(a_1) \wedge e_1) \vee \cdots \vee (h(a_n) \wedge e_n),$$

with $a_i \in \mathfrak{k}(L)$ and $e_i \in E(L)$ is compact. On the other hand, using the definition of compactness, along with (a), shows that any compact element of M can be so expressed. Once a compact element is in this form, thee e_i may be inductively disjointified, since $E(L)$ is a boolean algebra of complemented elements.

Finally, it has been mentioned in 2.8 that dM is a regular frame. The rest of (b) is easy, and is left to the reader. ∎

3 Spectra through p-Isomorphisms.

We turn to the topological view of the foregoing, the goal being to track the spectra of the frames linked by p-isomorphisms. The setting is, generally, the category $\mathfrak{CohFrm}$. Our ultimate goal is to compare the spectra of the frames involved when $h : L \longrightarrow M$ is a $\mathfrak{CohFrm}$-morphism which patch-generates M.

Up to this point we have not needed, and have avoided making any assumptions beyond the usual Zermelo-Fraenkel axioms. Having algebraic structures in mind for the applications to come, we now incorporate Zorn's Lemma into our assumptions. As is well known, this makes every algebraic frame *spatial*, in the sense that each member of the frame is a meet of primes.

Definition & Remarks 3.1. Throughout this commentary it is assumed that $h : L \longrightarrow M$ is an $\mathfrak{AFrm}$-morphism between the algebraic frames with the FIP L and M. The spectrum $\mathrm{Spec}(L)$ of a frame is considered as a topological space under the hull-kernel topology. To review, this is the topology whose open sets are the

$$\mathrm{coz}(x) \equiv \{\, p \in \mathrm{Spec}(L) \,:\, x \not\leq p \,\},$$

for each $x \in L$.

It is well known that the restriction of the adjoint h_* to $\mathrm{Spec}(M)$ maps into $\mathrm{Spec}(L)$, and that the resulting restriction $\mathrm{Spec}(h) : \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(L)$ is continuous. Further, if h is skeletal, then, applying the argument in the proof of Theorem 2.3 to the nucleus d, we get that h_*, restricted to dM is $d(h)_*$. Restricting to spectra, we have the commutative square

(3.1.1)
$$\begin{array}{ccc}
\mathrm{Spec}(L) & \xleftarrow{\ \mathrm{Spec}(h)\ } & \mathrm{Spec}(M) \\
{\scriptstyle (d_L)_*}\uparrow & & \uparrow{\scriptstyle (d_M)_*} \\
\mathrm{Spec}(dL) & \xleftarrow{\ \mathrm{Spec}(h)|_{\mathrm{Spec}(dM)}\ } & \mathrm{Spec}(dM)
\end{array}$$

Note that the vertical maps are inclusions.

When h is a p-isomorphism, then, according to Corollary 2.5, the induced $d(h)$ is one-to-one, making its adjoint surjective. What we wish to highlight is that $\mathrm{Spec}(d(h)) : \mathrm{Spec}(dM) \longrightarrow \mathrm{Spec}(dL)$ is also surjective. We record that in a lemma, which is well known; but we supply a sketch of the proof anyway.

Lemma 3.2. *Suppose that $h : L \longrightarrow M$ is a coherent embedding of algebraic frames. Then the induced map $\mathrm{Spec}(h) : \mathrm{Spec}(M) \longrightarrow \mathrm{Spec}(L)$ is surjective.*

Proof. Suppose that $p \in \mathrm{Spec}(L)$. Let S_p denote the set of all $y \in M$, such that $h_*(y) = p$. We noted that h_* is surjective, and so S_p is nonempty. We now show that Zorn's Lemma may be applied to S_p. For suppose that $T \subseteq S_p$ is a chain, and let $y = \bigvee T$; we argue that $h_*(y) = p$. If not so, there is a compact element $a \in L$ such that $a \leq h_*(y)$ but $a \not\leq p$. But then $h(a) \leq y$, and since $h(a)$ is compact in M, we conclude that $h(a) \leq t$ for some $t \in T$, which in turn implies that $a \leq h_*(t) = p$, a contradiction.

Thus, S_p possesses a maximal q; a routine check, using the fact that adjoints preserve meets, reveals that q must be prime in M. ∎

Here is the upshot of Lemmas 3.2 and 1.4, with regard to the discussion in 3.1.

Corollary 3.3. *Suppose that the* $\mathfrak{AFrm}$*-map* $h : L \longrightarrow M$ *is a* p*-isomorphism. Then the induced* $\mathrm{Spec}(d(h)) : \mathrm{Spec}(dM) \longrightarrow \mathrm{Spec}(dL)$ *is surjective.*

We narrow the discussion to consider the relationship between minimal spectra under a $\mathfrak{CohFrm}$-map which is a p-isomorphism. For any frame L, $\mathrm{Min}(L)$ stands for the subspace of minimal primes of L, under the relative hull-kernel topology. For any algebraic frame L, it is well known that every prime exceeds a minimal prime.

For an algebraic frame L with the FIP, the equivalence of conditions (a) and (b) in the lemma below is well known; see [M73, Corollary 2.5.1]. $P_\omega L$ denotes the subalgebra of polars generated by the polars of the form $a^{\perp\perp}$ $(a \in \mathfrak{k}(L))$.

Lemma 3.4. *Suppose that* L *is an algebraic frame with the FIP. Then the following are equivalent for* $p \in \mathrm{Spec}(L)$*.*

(a) $p \in \mathrm{Min}(L)$.

(b) $p = \vee \{ a^\perp \,:\, a \not\leq p,\ a \in \mathfrak{k}(L) \}$.

(c) *For each* $x \in P_\omega L$*, either* $x \leq p$ *or* $x^\perp \leq p$*, but not both.*

Moreover, $\mathrm{Min}(L)$ *is a Hausdorff space with a base of clopen sets.*

Proof. As a consequence of (b), p exceeds exactly one of $a^{\perp\perp}$ and $a^\perp$; this is straightforward. Thus, (b) is a special case of (c). Now assume (b). Note that each $x \in P_\omega L$ can be expressed as a finite supremum of infima of polars of the form $a^{\perp\perp}$ and $b^\perp$, with a and b compact. Put

$$x = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{k_i} x_{i,j},$$

where each $x_{i,j}$ is either a $a^{\perp\perp}$ (with $a \in \mathfrak{k}(L)$) or else the complement of one. Now, $x \leq p$ precisely when each $\wedge_{j=1}^{k_i} x_{i,j} \leq p$, which in turn occurs if and only if, for each $i = 1, \ldots, n$, there is a $j = 1, \ldots, k_i$, such that $x_{i,j} \leq p$. But

$$x^\perp = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{k_i} x_{i.j}^\perp.$$

Continuing, $x \leq p$ if and only if, for each $i = 1, \ldots, n$, there is a $j = 1, \ldots, k_i$ such that $x_{i,j}^\perp \not\leq p$, which is so precisely when each $\vee_{j=1}^{k_i} x_{i,j}^\perp \not\leq p$, and therefore if and only if $x^\perp \not\leq p$.

Let $\mathrm{coz}_{\mathrm{Min}}(x) \equiv \mathrm{Min}(L) \cap \mathrm{coz}(x)$, for each $x \in L$; this is a typical open set of $\mathrm{Min}(L)$. The subfamily of all $\mathrm{coz}_{\mathrm{Min}}(a)$, with a compact, form a base for the open sets; they are also closed, by the foregoing, as $\mathrm{coz}_{\mathrm{Min}}(a^\perp) = \mathrm{Min}(L) \setminus \mathrm{coz}_{\mathrm{Min}}(a)$, for each $a \in \mathfrak{k}(L)$. That $\mathrm{Min}(L)$ is Hausdorff is now obvious. ∎

We have the resources now to examine a $\mathfrak{CohFrm}$-map $h : L \longrightarrow M$ which patch-generates M. First, let us observe some special features of minimal primes Part (a) of the following lemma is part of [MZ03, Theorem 2.4]. Condition (b) is an immediate consequence of the preceding lemma, and (d) is also well known. As to (c), note that since, dL is regular, for any projectable frame, $\mathrm{Min}(L) = \mathrm{Spec}(dL)$ ([MZ03, Theorem 2.4]).

Lemma 3.5. *Let L be an algebraic frame with the FIP. Then*

(a) *Each minimal prime is a d-element.*

(b) *If p and q are distinct minimal primes of L and L is projectable, then $p \vee q = 1$. Thus, each prime exceeds a unique minimal prime.*

(c) *If L is a coherent projectable frame then $\mathrm{Min}(L) = \mathrm{Spec}(dL)$ is a compact Hausdorff space.*

Remark 3.6. Suppose that $h : L \longrightarrow M$ is a d-isomorphism. As noted in Proposition 2.6, for each compact $b \in M$, there is a compact $a \in L$ such that $b^{\perp\perp} = h(a)^{\perp\perp}$. The condition in turn implies that h is a p-isomorphism. Then, using Lemma 3.4, one can show that $\mathrm{Spec}(h)$ induces a homeomorphism of $\mathrm{Min}(M)$ onto $\mathrm{Min}(L)$. Compare with [CM90, Proposition 2.3].

The following result is interesting. Since we shall not use it anywhere in this paper, we merely sketch the proof.

Proposition 3.7. *Suppose that $h : L \longrightarrow M$ is a $\mathfrak{CohFrm}$-morphism which patch-generates M. If $p < q$ in $\mathrm{Spec}(M)$, then $\mathrm{Spec}(h)(p) < \mathrm{Spec}(h)(q)$.*

Proof. That $\mathrm{Spec}(h)(p) \leq \mathrm{Spec}(h)(q)$ is clear. Observe next that p and q exceed the same complemented elements. So now consider a $b \in \mathfrak{k}(M)$, expressed as in Proposition 2.9(a),

$$(h(a_1) \wedge e_1) \vee \cdots \vee (h(a_n) \wedge e_n),$$

where each $a_i \in \mathfrak{k}(L)$ and each $e_i \in E(L)$, with the $e_1, \ldots, e_n$ pairwise disjoint.

Suppose that $b \leq q$, but not beneath p. Note that each $h(a_i) \wedge e_i \leq q$, while exactly one $h(a_k) \wedge e_k \not\leq p$, and thus $e_k \not\leq p$, while the other $e_i \leq p$. Note as well that $h(a_k) \not\leq p$, whence $a_k \not\leq \mathrm{Spec}(h)(p)$. On the other hand, since $e_k \not\leq q$, $h(a_k) \leq q$, and so $a_k \leq \mathrm{Spec}(h)(q)$. Then a_k witnesses that $\mathrm{Spec}(h)(p) \neq \mathrm{Spec}(h)(q)$. ■

Remark 3.8. Let L be a coherent frame. By [MZ03, Corollary 2.6], L is regular precisely when each compact element has a compact complement. For dL the latter amounts to the following ([MZ03, Proposition 5.2]): *For each $a \in \mathfrak{k}(L)$ there is a $b \in \mathfrak{k}(L)$ such that $a \wedge b = 0$ and $a \vee b$ is dense.*

If this is the case, then $\mathrm{Spec}(dL) = \mathrm{Min}(L)$ and compact Hausdorff space. Projectable frames are a special case of this more general situation.

We are almost to the main theorems of the section. What is still needed, for a map h which patch-generates M, is a way to map the minimal primes of L to those of M. This is what the next lemma provides.

The reader would do well to review [M73, Lemma 2.5] or [MZ03, Lemma 2.2], on the relationship between ultrafilters of compact elements.

Lemma 3.9. *Suppose that the* $\mathfrak{CohFrm}$*-map* $h : L \longrightarrow M$ *patch-generates* M*. Let* $n \in \mathrm{Min}(L)$ *and* U *be an ultrafilter of compact elements of* M*, containing*

$$S_n \equiv \{\, h(a) \,:\, a \not\leq n,\ a \in \mathfrak{k}(L)\,\}.$$

Define

$$\widetilde{n} \equiv \bigvee \Big\{\, b^{\perp} \,:\, b \in U \,\Big\}. \tag{3.9.1}$$

Then $\widetilde{n}$ *is a minimal prime of* M *such that* $\mathrm{Spec}(h)(\widetilde{n}) = n$.

Conversely, if $p, q \in \mathrm{Min}(M)$ *and* $\mathrm{Spec}(h)(p) = \mathrm{Spec}(h)(q)$*, and lies in* $\mathrm{Min}(L)$*, then* $\mathfrak{p} = \mathfrak{q}$*. Thus, the definition of* $\widetilde{n}$ *is independent of the choice of* U*, and the mapping* $n \mapsto \widetilde{n}$ *is one-to-one.*

Proof. First, it is easy to see that S_n is a filter of compact elements of M, and Zorn's Lemma furnishes us with an ultrafilter U containing S_n. That $\widetilde{n}$ is a minimal prime follows from [M73, Lemma 2.5]. Further, since a minimal prime is a d-element < 1, by Lemma 3.5(a), we have that $n \leq \mathrm{Spec}(h)(\widetilde{n})$. Were the inequality to be strict, we would have a compact $a \in L$ such that $a \not\leq n$, yet $a \leq \mathrm{Spec}(h)(\widetilde{n}) = h_*(\widetilde{n})$, so that $h(a) \leq \widetilde{n}$, which is absurd.

Suppose that $p, q \in \mathrm{Min}(M)$ and $\mathrm{Spec}(h)(p) = \mathrm{Spec}(h)(q)$, and is minimal. Then, for each $a \in \mathfrak{k}(L)$, $a^{\perp\perp}$ or its polar is below $\mathrm{Spec}(h)(p)$ if and only if it is below $\mathrm{Spec}(h)(q)$, and therefore, by Lemma 3.4, $\mathrm{Spec}(h)(p)$ and $\mathrm{Spec}(h)(q)$ exceed the same elements of $P_\omega L$. Thus (applying $P(h)$), p and q exceed the same elements of $E(L)$. Invoking Proposition 2.9, p and q exceed the same compact elements, and therefore $p = q$.

The final claim is then obvious. ∎

The following is one of the principal results of this section; the notation in the theorem is the same as that of the preceding lemma. Let $\widetilde{L}$ denote the set of all $p \in \mathrm{Min}(M)$ such that $\mathrm{Spec}(h)(p)$ is a minimal prime of L.

Theorem 3.10. *Suppose that the* $\mathfrak{CohFrm}$*-map* $h : L \longrightarrow M$ *patch-generates* M*. Then we have:*

(a) *For each* $p \in \widetilde{L}$, $p = \widetilde{\mathrm{Spec}(h)(p)}$*, so that* $n \mapsto \widetilde{n}$ *is inverse to the restriction of* $\mathrm{Spec}(h)$ *to* $\widetilde{L}$*; this restriction witnesses that* $\widetilde{L}$ *is homeomorphic to* $\mathrm{Min}(L)$*.*

(b) $\widetilde{L}$ *is a dense subspace of* Min(M). *Thus,* Min(M) *is (homeomorphic to) a compactification of* Min(L).

Proof. (a) Suppose that $p \in \widetilde{L}$. Note at the outset that $a \in \mathfrak{k}(L)$ satisfies $a \not\leq \mathrm{Spec}(h)(p) = h_*(p)$ if and only if $h(a) \not\leq p$. By Lemma 3.9 (and, specifically, the independence upon the ultrafilter in the definition of $\widetilde{(\cdot)}$), $p = \widetilde{\mathrm{Spec}(h)(p)}$. The restriction of Spec(h) to $\widetilde{L}$ then maps it continuously and bijectively onto Min(L). The reader may easily check that the map is open.

(b) To see that $\widetilde{L}$ is dense, we must show that $y \equiv \bigwedge \widetilde{L} = 0$. But,

$$h_*(y) = \bigwedge \left\{ h_*(p) \,:\, p \in \widetilde{L} \right\} = \bigwedge \mathrm{Min}(L) = 0,$$

and since h is $*$-dense, we may conclude that $y = 0$. Then, by Lemma 3.5(c), Min(M) is a compactification of Min(L). ■

The compactification of Theorem 3.10(b) need not be the Stone-Čech compactification; see the examples in §9.

Remark 3.11. We shall presently improve upon Theorem 3.10. Assuming the same setup and notation as in the theorem and Lemma 3.9, it should be evident that in the definition of $n \mapsto \widetilde{n}$ (3.9.1), the assumption that n is a minimal prime does not play a role. In the proof of the identity $n = \mathrm{Spec}(h)(\widetilde{n})$ for Lemma 3.9 all that was needed was that $n \in \mathrm{Spec}(dL)$. Thus, if we can establish that Spec(h) is one-to-one on Min(M), we will have shown that Spec(dL) and Min(M) are in a bijective correspondence, by $\mathrm{Spec}(h)|_{\mathrm{Min}(M)}$, with inverse $n \mapsto \widetilde{n}$.

Lemma 3.12. *Suppose that the* $\mathfrak{CohFrm}$*-map* $h : L \longrightarrow M$ *patch-generates* M. *Then* $n \mapsto \widetilde{n}$ *is well defined on* Spec(dL), *and is a bijection onto* Min(M), *with inverse* $\mathrm{Spec}(h)|_{\mathrm{Min}(M)}$.

Proof. As shown in the proof of of Lemma 3.9, we have $n = \mathrm{Spec}(h)(\widetilde{n})$, for each $n \in \mathrm{Spec}(dL)$. As before, this shows that the definition of $\widetilde{n}$ does not depend on the choice of the ultrafilter.

To reverse, let $q \in \mathrm{Min}(M)$, and consider $S_q = \{\, b \in \mathfrak{k}(M) \,:\, b \not\leq q \,\}$; let T be the set of complemented elements in S_q. Then T is an ultrafilter of complemented elements of M; recall that, per Proposition 2.9(b), $e \in M$ is complemented if and only if $e \in \mathfrak{k}(dM)$. To prove this claim, we first observe that T is evidently a filter. So suppose that $f \in \mathfrak{k}(dM)$, and that $f \wedge e \neq 0$, for each $e \in T$. Now, if $b \in S_q$, and $e_b = b^{\perp\perp}$, then $b = b \wedge e_b$, whence $e_b \in T$ and $f \wedge b \neq 0$. Since S_q is an ultrafilter of $\mathfrak{k}(M)$, we have that $f \in T$, proving the claim.

The following is also easy to check: that $e \in T$ precisely when $e = a^{\perp}$, for some compact $a \leq q$.

Next, let

$$p \equiv \bigvee \left\{ h_*(f^\perp) \,:\, f \in T \right\}.$$

Then $p \in \mathrm{Spec}(dL)$, and it is routine to verify that $p = \mathrm{Spec}(h)(q)$; we leave this to the reader as well.

To conclude, note that $b \in \mathfrak{k}(dM)$ lies beneath q if and only if $e = b^\perp \notin T$, that is to say, $e = b^\perp \nleq q$. Since $\widetilde{p}$ is independent of the ultrafilter used to define it (and $p = \mathrm{Spec}(h)(q)$), it follows that $b \leq q$ if and only if $b \leq \widetilde{p}$, and $q = \widetilde{p}$. This suffices to prove the lemma. ∎

The final result in this section identifies $\mathrm{Spec}(dM) = \mathrm{Min}(M)$ as being homeomorphic to $\mathrm{Spec}(dL)$, with the finer patch topology. Let us remind the reader briefly of this refinement.

Definition 3.13. Let X be a topological space, with a base $\mathcal{B}$ for the open sets of X. The *$\mathcal{B}$-patch topology* on X is the (finer) topology having as base the members of the boolean subalgebra of the power set of X generated by $\mathcal{B}$. As it applies to $\mathrm{Spec}(L)$ of an algebraic frame L with the FIP, we shall in this article always use the patch topology generated by the base of sets lying in the subalgebra generated by the $\mathrm{coz}(a)$, with a compact. Since L has the FIP, the sets of the form

$$\mathrm{coz}(a) \cap \mathrm{coz}(b^\perp), \quad a, b \in \mathfrak{k}(L),$$

are a base for the patch topology .

With all that has gone before the final result of the section will only require a very brief comment by way of proof.

Theorem 3.14. *Suppose that the* $\mathfrak{CohFrm}$*-map* $h : L \longrightarrow M$ *patch-generates* M. *Then* $n \mapsto \widetilde{n}$ *is a homeomorphism of* $\mathrm{Spec}(dL)$ *with the patch topology onto* $\mathrm{Min}(M)$.

Proof. For each $p \in \mathrm{Spec}(dL)$, we have $p = \mathrm{Spec}(h)(\widetilde{p}) = h_*(\widetilde{p})$. Then observe that, for each compact $a \in L$, $a^{\perp\perp} \leq p$ precisely when $h(a)^{\perp\perp} \leq \widetilde{p}$. This, together with the fact that for $\mathrm{Min}(M)$ the hull-kernel topology and patch topology coincide, and Lemma 3.12, proves this theorem. ∎

We shall refer to the map $n \mapsto \widetilde{n}$ as the *d-magnification* of primes; it will be denoted μ_d. The first two examples of §9 show that μ_d differs from $d(h)$.

Remark 3.15. The referee of this paper posed an interesting question: *Does there exist, for each coherent frame L, a* $\mathfrak{CohFrm}$*-map $h : L \longrightarrow M$ that patch-generates M, and is it unique?*

We do not know. In this work we have concentrated on the relationship between the structures of the two frames involved. Regarding the question of existence, we have made some progress on a related question, namely, the existence of a strongly projectable hull; let us refer the reader to [MZ06c].

As the reader will see, presently, a patch-generating map need not be a frame embedding. In fact, it is quite frequently a nontrivial quotient.

4 Extending and Tracing.

We turn to the projectable hull of commutative rings. All rings in this article are commutative with identity, and, indeed, *semiprime*; that is, they have no nonzero nilpotent elements. The foregoing frame theory is applied to the frame of radical ideals, about which we now review the basics, referring the reader to [M06] for details.

Definition & Remarks 4.1. Let A be a ring. An ideal $\mathfrak{r}$ of A is a *radical* ideal if $a^2 \in \mathfrak{r} \;\Rightarrow\; a \in \mathfrak{r}$. $\mathrm{Rad}(A)$ denotes the set of all radical ideals. As is well known, an ideal is radical if and only if it is an intersection of prime ideals. $\mathrm{Rad}(A)$ is a coherent frame, and, by a theorem of Hochster – see [Ho69] or [Ba96] – every coherent frame arises as $\mathrm{Rad}(A)$ for a suitable ring A.

The closure operator associated with $\mathrm{Rad}(A)$ is defined on the lattice of all ideals by

$$\mathfrak{r} \mapsto \sqrt{\mathfrak{r}} \equiv \{\, a \in A \,:\, a^k \in \mathfrak{r},\ \text{for suitable } k \in \mathbb{N} \,\}.$$

$\sqrt{\mathfrak{r}}$ is also the intersection of all prime ideals of A containing $\mathfrak{r}$.

Now suppose $A \le B$ is a ring extension. Suppose that $\mathfrak{r} \in \mathrm{Rad}(A)$. Let $\varepsilon(\mathfrak{r}) = \sqrt{B\mathfrak{r}}$; it is easily checked that $\varepsilon(\mathfrak{r})$ is the least radical ideal of B containing $\mathfrak{r}$. ε is a dense coherent frame map. Moreover,

- the adjoint $\varepsilon_* = \tau$, where $\tau(\mathfrak{s}) = \mathfrak{s} \cap A$, for each $\mathfrak{s} \in \mathrm{Rad}(B)$;
- put $j_{\mathrm{rad}} = \tau \cdot \varepsilon$; this is a nucleus, and $\mathfrak{r} = j_{\mathrm{rad}}(\mathfrak{r})$ if and only if

$$a^k = \sum_{i=1}^{m} b_i r_i, \quad \text{with} \quad b_i \in B,\ r_i \in \mathfrak{r},\ \Longrightarrow\ a \in \mathfrak{r};$$

ε and τ will be referred to as *extension* and *trace*, respectively.

Definition & Remarks 4.2. Let A be a ring.

(a) The polars of $\mathrm{Rad}(A)$ are none other than the annihilators of A; recall that $\mathfrak{r}$ is an *annihilator* if it is of the form $\mathfrak{r} = \{\, x \in A \,:\, xS = \{0\},\ \text{for some } S \subseteq A \,\}$. We denote the boolean algebra of annihilators $\mathcal{P}(A)$. For $x \in A$, we will write $x^\perp \equiv (\sqrt{Ax})^\perp$.

(b) A is *von Neumann regular* if, for each $a \in A$ there is an $x \in A$ such that $a^2x = a$. Letting $y = x^2a$, then we also have $a^2y = a$ and $y^2a = y$. Note that y is unique with respect to the latter two identities; this is the *quasi-inverse* of a. Note further that

$ay = e$ is an idempotent, that $Ae = Aa$, and that e is the unique idempotent generator for the ideal Aa. We shall say that e is *the idempotent of* a. Moreover, as is well known – see [L86] – A is von Neumann regular if and only if each principal ideal is generated by an idempotent.

It is immediate that A is von Neumann regular if and only if $\mathrm{Rad}(A)$ is a regular frame ([MZ03, 7.1]).

(c) We shall call A *projectable* if $\mathrm{Rad}(A)$ is a projectable frame. Thus, A is projectable if and only if

$$A = a^{\perp\perp} \oplus a^{\perp}, \quad \text{for each } a \in A.$$

Evidently, every von Neumann regular ring is projectable.

The converse is false. Consider, for instance, the ring A of bounded real valued functions on an uncountable set D which are constant on all but a countable subset of D. It is easy to verify that A is a projectable ring, which is not von Neumann regular; for example, let $S = \{d_1, \ldots, d_n, \ldots\}$ be a countable subset of D. Define $f \in A$ by $f(d) = 0$, if $d \notin S$, while $f(d_n) = \frac{1}{n}$. A quasi-inverse of f would have to be unbounded.

Next, we consider the notion of a hull and its associated hull class.

Definition 4.3. A class $\mathfrak{H}$ of rings is said to be a *hull class* if for each ring A there is an A-*essential* extension hA of A in $\mathfrak{H}$, such that if $A \leq B$ is an A-essential extension, and $B \in \mathfrak{H}$, then the inclusion extends to a one-to-one homomorphism $f : hA \longrightarrow B$. hA is the $\mathfrak{H}$-*hull* of A.

The term "A-essential" is interpreted in the obvious way: a ring extension B of A is also an A-module. Thus, $A \leq B$ is A-*essential* if and only if $b \neq 0$ in B implies that $ba \neq 0$ and an element of A, for suitable $a \in A$. Note that $A \leq B$ is A-essential precisely when the trace $\tau : \mathrm{Rad}(B) \longrightarrow \mathrm{Rad}(A)$ is dense. Thus, if the ring extension is A-essential, then ε is dense and $*$-dense, and hence all the resources of Sections 2 and 3 may be applied.

The general references for rings of quotients are [L86], and for the latter half of the paper, [M95]. The generalized concept of "ring of quotients" is due to Utumi ([U56]).

Definition & Remarks 4.4. Again, suppose that A is a ring.

(a) An extension B of A is called a *ring of quotients* provided, for each $b_1, b_2 \in B$, with $b_2 \neq 0$, there is an $a \in A$ such that $ab_1 \in A$ and $ab_2 \neq 0$. Define the ideal of A

$$b^{-1}A \equiv \{\, a \in A \,:\, ab \in A \,\}.$$

Then B is a ring of quotients of A if and only if each $b^{-1}A$ is dense in B, in the sense that if $c \in B$ and $c(b^{-1}A) = \{0\}$, then $c = 0$. This is well known and easy to verify.

There is a maximum ring of quotients, denoted QA. It is easy to see that $A \leq QA$ is an A-essential extension. In fact, it is a consequence of [L86, 2.2,§4.5], that QA is the largest A-essential extension of A; that is to say, it is the injective hull of A, over

A. Notice as well, that any extension of A which is a subring of QA is A-essential, so that the A-essential extensions of A and its rings of quotients coincide.

(b) qA stands for the *classical ring of quotients*. With reference to QA it may be regarded as

$$qA = \{\, b \in QA \,:\, b^{-1}A \text{ contains a regular element} \,\}.$$

Alternatively, and as is well known, the elements of qA may then be viewed as formal fractions a/d, with $a, d \in A$, where d is regular in A. (Note: "regular element"$\equiv$ "non zero divisor".) Equality of fractions is defined in the familiar way, as is the arithmetic.

Note that the class of rings for which $qA = A$, that is, the class of rings in which every regular element is invertible, is a hull class, and that qA is the hull of A in this class. This is well known, and is easily established by observing that if $d \in A$ is regular in A, and B is a ring of quotients of A, then d is also regular in B.

Now we have the following characterization of hull classes of rings. The proof is straightforward, and is left to the reader.

Proposition 4.5. *Suppose that $\mathfrak{H}$ is a class of rings, closed under formation of isomorphic copies. Then $\mathfrak{H}$ is a hull class precisely when the following holds: for each ring A and any family $\{\, B_\lambda \,:\, \lambda \in \Lambda \,\}$ of rings of quotients of A, all belonging to $\mathfrak{H}$, we have that $\cap_\lambda B_\lambda \in \mathfrak{H}$.*

Definition 4.6. Borrowing the terminology from [HM99], let us say that a class of rings is *essentially intersective* if it satisfies the condition of Proposition 4.5.

Remark 4.7. We recommend that the reader refer to [M02] often in the next section. It is reasonably self-contained; however, since the proofs of corresponding material carry over to rings *mutatis mutandis*, we will omit most of them, leaving it to the reader to look them up in [M02], as needed.

5 The Projectable Hull.

Throughout, A denotes a semiprime ring. We begin with a comment about idempotents vs annihilators. We denote $P_\omega\mathrm{Rad}(A)$ by $\mathcal{P}_\omega(A)$; this is the subalgebra of annihilators generated by the annihilators of the form $f^{\perp\perp}$, for all $f \in A$.

Definition & Remarks 5.1. The complemented elements of $\mathrm{Rad}(A)$ are just the ideals which are cardinal summands. In turn these are the ideals of the form $e^{\perp\perp}$, for a suitable idempotent e.

It is well known that in QA every annihilator is a summand; see [L86, Proposition 4, §2.4]. If $K \in \mathcal{P}(A)$, then let us denote the projection of 1 on $K^{\perp\perp}$ (in QA) by $e(K)$. Denote by $E_\omega(A)$ the set of idempotents of the form $e(K)$, for all $K \in \mathcal{P}_\omega(A)$. It should be clear that $E_\omega(A)$ is a boolean algebra of idempotents, isomorphic to $\mathcal{P}_\omega(A)$, by the map $K \mapsto e(K)$.

We begin with a lemma, which is routine, and is left to the reader. The details mimic [M02, Lemma 2.2].

Lemma 5.2. *Let B be a ring of quotients of the semiprime ring A. Then B is projectable if and only if, for each $K \in \mathcal{P}_\omega(A)$, $e(K) \in B$.*

The next proposition establishes the existence of the projectable hull in (a); the preceding lemma takes care of that. The proof of (b) is essentially that of [M02, Theorem 2.4], but we shall adapt it here anyway, for completeness. Compare also with the proof of Proposition 2.9.

Proposition 5.3. *Let A be a semiprime ring.*

(a) *There is a least projectable extension pA of A.*

(b) *The elements of pA are the expressions of the form*

$$\sum_{i=1}^{n} a_i e_i, \tag{5.3.1}$$

with $a_i \in A$, and $e_i = e(K_i)$, for suitable annihilators $K_i \in \mathcal{P}_\omega(A)$; the e_i may be chosen so that $e_i e_j = 0$, for $i \neq j$.

Proof. (b) Let A' be the subset of all expressions (5.3.1); it should be clear that A' is a subring of QA. By Lemma 5.2, every projectable extension contains A', whence $A' \leq pA$. To prove the reverse containment, it suffices to show that A' is projectable. As to that, suppose that $K \in \mathcal{P}_\omega(A)$; then $e(K) \in A'$, which immediately implies that

$$A' = e(K)^{\perp\perp} \oplus e(K)^{\perp} = K^{\perp\perp} \oplus K^{\perp}.$$

Finally, we establish that every expression (5.3.1) can be rewritten with $e_i e_j = 0$, for all $i \neq j$. We do this for two idempotents and allow the reader to supply the induction. Suppose that $a, b \in A$ and $K, L \in \mathcal{P}_\omega(A)$; consider $x = ae(K) + be(L)$. The reader should note that $e(K) = e(K \cap L) + e(K \cap L^{\perp})$; a similar equation holds for $e(L)$. We calculate:

$$x = ae(K) + be(L) = (a + b)e(K \cap L) + ae(K \cap L^{\perp}) + be(K \cap K^{\perp}).$$

This disjointifies the expression for x. ∎

The disjointification of idempotents comes in handy for the following consequence of Proposition 5.3; it will be needed again in §7.

Corollary 5.4. *Suppose that A is a semiprime ring. Then $E_{\mathcal{P}_\omega}(A)$ is the boolean algebra of all idempotents of pA.*

Proof. It is clear that each member of $E_{\mathcal{P}_\omega}(A)$ lies in pA. Now suppose that $x = x^2 \in pA$, and write $x = \sum_{i=1}^n a_i e_i$, as specified in (5.3.1), with $e_i e_j = 0$, whenever $i \neq j$. It is clear that each $a_i e_i$ must be idempotent. For each $i = 1, \ldots, n$, let $K_i \in \mathcal{P}_\omega(A)$, such that $e_i = e(K_i)$. We leave it to the reader to verify that $a_i e_i = e(a_i^{\perp\perp} \cap K_i)$, and that $x = e(\vee_{i=1}^n a_i^{\perp\perp} \cap K_i)$. ■

Remark 5.5. We use $\mathrm{Min}(A)$ for $\mathrm{Min}(\mathrm{Rad}(A))$. We have seen that this is a Hausdorff space with a base of clopen sets. When A is projectable then $\mathrm{Min}(A)$ is also compact.

(a) We say that A is *complemented* if for each $a \in A$ there is a $b \in A$ such that $ab = 0$ and $a + b$ is regular. It is this condition which characterizes the semiprime rings A for which $\mathrm{Min}(A)$ is compact. This result first appeared in [HJ76]. We also mention [CM90], which proves a similar result, in the context of lattice-ordered groups.

In [M95, Proposition 1.2] it is shown that A is complemented if and only if qA is von Neumann regular. The context there is semiprime f-rings, but the result we have referred to has nothing to do with f-rings. We shall use it in the next section.

(b) A few comments regarding $d\mathrm{Rad}(A)$ are due here. Its members will be referred to as *d-ideals*. Obviously, no proper d-ideal may contain a regular element. It is not true that an ideal which fails to contain any regular elements is a d-ideal; see Example 9.1. However, if A is complemented, then every prime ideal without regular elements is a d-ideal ([M06, 3.3(a)]), and, in fact, minimal.

Evidently, each projectable ring is complemented.

(c) Moreover, if compact, $\mathrm{Min}(A)$ is the Stone dual of the boolean algebra of all idempotents of A, denoted $\mathcal{E}(A)$. The latter is canonically isomorphic to the boolean algebra of all summands $\mathcal{S}(A)$ of A; the isomorphism is $e \mapsto Ae$. Note that two minimal prime ideals of a projectable ring A are the same if and only if they contain the same idempotents.

We highlight the comments in 5.5(c), in an immediate consequence of Corollary 5.4.

Corollary 5.6. *Let A be a ring. Then $\mathrm{Min}(pA)$ is the Stone dual of $E_{\mathcal{P}_\omega}(A) \cong \mathcal{P}_\omega(A)$.*

Let us denote the space $\mathrm{Spec}(d\mathrm{Rad}(A))$ by the shorter $\mathrm{Spec}_d(A)$. Putting Theorems 3.10 and 3.14 together with Proposition 5.3, we get the following.

Theorem 5.7. *Let A be a ring. Then the d-magnification μ_d is a homeomorphism from $\mathrm{Spec}_d(A)$ with the patch topology onto $\mathrm{Min}(pA)$, whose inverse is the trace map. μ_d induces a dense embedding of $\mathrm{Min}(A)$, with the hull-kernel topology, in $\mathrm{Min}(pA)$.*

6 The Regular Hull.

In this brief section we collect some information about the regular hull of a semiprime ring, based on what we now know about the smaller projectable hull. The existence of the regular hull is not at issue; several authors have already written extensively about it. The reader is referred to [RW99] and [St68]. We shall sketch, for the sake of completeness, how existence follows from Proposition 4.5, by showing that the class of von Neumann regular rings is essentially intersective.

Proposition 6.1. *The class of von Neumann regular rings is essentially intersective. Thus, each semiprime ring A has a von Neumann regular hull HA.*

Proof. Let $a \in A$, and a^* be the quasi-inverse of a in QA (4.2(b)). Since von Neumann regular rings are projectable, Lemma 5.2 guarantees that e, the idempotent of a, belongs to each von Neumann regular ring of quotients of A. Now suppose that B is such a ring of quotients, and let $b \in B$ be the quasi-inverse of a in B; then

$$b = be = baa^* = ea^* = a^*,$$

which proves that if $\{ B_\lambda : \lambda \in \Lambda \}$ is a family of von Neumann rings of quotients of A, then their intersection must be one as well. Now apply Proposition 4.5. ■

From the material of the previous section we recover the description of HA of [RW99, Lemma 2.4].

Proposition 6.2. *Let A be a semiprime ring. Then $HA = qpA$.*

Proof. Since pA is projectable, it follows that qpA is von Neumann regular (5.5(a)). On the other hand, if B is a von Neumann regular ring of quotients, it follows that $pA \leq B$, and since $qB = B$, we conclude that $qpA \leq B$. ■

Remark 6.3. Proposition 6.2 prompts the question of whether the application of the projectable hull *after* the classical ring of quotients also yields the regular hull. The answer is no. We give an example in the next section.

Remark 6.4. Let us return to the frame-theoretic context for a moment, and consider the extension ε from $\mathrm{Rad}(A)$ to $\mathrm{Rad}(qA)$. It was noted in [M06, Theorem 7.6] that, in this case, ε is surjective. By the comment in 1.5, $d(\varepsilon) : d\mathrm{Rad}(A) \longrightarrow d\mathrm{Rad}(qA)$ is an isomorphism of frames, which carries $\mathrm{Min}(A)$ onto $\mathrm{Min}(qA)$, and witnesses that they are homeomorphic.

On the other hand, as is well known, every homomorphic image of a von Neumann regular ring is von Neumann regular. In particular, every ideal of HA is a radical ideal, which means that $d\mathrm{Rad}(HA) = \mathrm{Rad}(HA) = \mathcal{I}(HA)$, the latter being the frame of all ideals of HA.

Putting these thoughts together, we can now formulate a version of Theorem 5.7 for the passage from A to HA.

Theorem 6.5. *Let A be a ring. Then we have the following:*

(a) *The trace map $\tau : \mathcal{I}(HA) \longrightarrow d\mathrm{Rad}(A)$ is surjective.*

(b) *$\tau|_{\mathrm{Min}(HA)} : \mathrm{Min}(HA) \longrightarrow \mathrm{Spec}_d(A)$ is a homeomorphism onto $\mathrm{Spec}_d(A)$ with the patch topology. The inverse map is $\varepsilon \cdot \mu_d$.*

(c) *$\varepsilon \cdot \mu_d$ induces a dense embedding of $\mathrm{Min}(A)$, with the hull-kernel topology, in $\mathrm{Min}(HA)$.*

7 f-Rings.

We turn now to f-rings. The standard assumption will be that, unless otherwise noted, "f-ring" means "commutative semiprime f-ring with identity". To recall, an f-ring A is a lattice-ordered group with respect to addition, and in addition $a \wedge b = 0$ and $c \geq 0$ imply that $a \wedge bc = 0$. For the very basic information on f-rings the reader is referred to [BKW77]. We begin by noting, more precisely, some of the common facts regarding distinguished ideals of an f-ring.

Definition & Remarks 7.1. Let A be an f-ring. The following are all well known.

(a) *Every minimal prime ideal is convex, and therefore an ℓ-ideal.*

"Convex" here means order-convex. An ℓ-ideal is a ring ideal which is convex and also an *ℓ-subgroup*; that is to say, closed under the lattice operations.

Indeed $\mathrm{Min}(A)$ coincides with the set of convex ℓ-subgroups P which are prime in the sense that $a \wedge b = 0$ implies that $a \in P$ or $b \in P$, and minimal among these ([BKW77, Theorem 9.3.2]).

(b) *For each $S \subseteq A$,*

$$S^{\perp} = \{\, a \in A \,:\, |a| \wedge |s| = 0 \,\}.$$

Put it another way, $|a| \wedge |b| = 0$ if and only if $ab = 0$ ([BKW77, Theorem 9.3.1]).

(c) *A is a subdirect product of totally ordered integral domains.*

Technically, Zorn's Lemma is required. On the other hand, if A is a subdirect product of totally ordered integral domains, the f-ring feature may be directly verified ([BKW77, Theorem 9.3.1]).

(d) *An archimedean f-ring with identity is semiprime.*

Note: A is *archimedean* if for each $0 \leq a, b \in A$, $na \leq b$, for each natural number n, implies that $a = 0$ ([BKW77, Theorem 12.3.9]).

Some conventions for the material in this section and the next are in order here.

Definition & Remarks 7.2. In this section the category of discourse is $\mathfrak{Spf}$, the category of all commutative semiprime f-rings with identity and all ring ℓ-homomorphisms that preserve the identity. $\mathfrak{Arf}$ denotes the full subcategory of all archimedean f-rings.

As is common, $C(X)$ denotes the ring of all continuous real valued functions on the topological space X, with respect to pointwise operations. $C(X)$ is an $\mathfrak{Arf}$-object. As is also the custom, without loss of generality, in dealing with $C(X)$, X is assumed to be a *Tychonoff space*; that is, X is Hausdorff, and for each point $p \in X$ and each closed set $C \subseteq X$ not containing p, there is an $f \in C(X)$ such that $f(p) = 0$ and $f(C) = \{1\}$. For any unexplained terminology regarding rings of continuous functions, the reader is referred to [GJ76].

βX stands for the Stone-Čech compactification of X. Recall that for each compact space X,

$$p \mapsto M_p \equiv \{\, f \in C(X) \,:\, f(p) = 0 \,\}$$

defines a homeomorphism of X onto $\mathrm{Max}(C(X))$, the space of maximal ideals of $C(X)$ ([GJ76, Theorem 4.11]).

Next, here are some basic observations regarding the rings of quotients of an f-ring.

Remark 7.3. There is a number of articles showing how the maximum ring of quotients, QA, of an f-ring A, may be lattice-ordered so as to make QA an f-ring, containing A as an ℓ-subring. We refer the reader to [An65, Ba65, M95], and, for rings of functions, to [FGL05]. An accomplishment of [Ba65] is to view QA as a direct limit of rings of functions, thus making it easier to see how the lattice-ordering of A may be extended naturally to QA ([M95]). If A is archimedean, then QA is as well; more will be said about that in the sequel.

qA is then an intermediate ℓ-subring. One can say directly when a fraction is positive: first, every $x \in qA$ may be written as $x = \frac{a}{d}$, with $d > 0$; that being given, $x \geq 0$ if and only $a \geq 0$.

The next item of business is to show that the projectable hull is an ℓ-subring of the maximum ring of quotients. Please keep in mind that the fact that the projectable hull of an f-ring is an f-ring is not new. However, that it is embedded as an ℓ-subring of QA has, to our knowledge, been overlooked.

Proposition 7.4. *Suppose that A is an f-ring. Then pA is an ℓ-subring of QA.*

Proof. We use Proposition 5.3(b). If $x \in pA$, then we may write

$$x = \sum_{i=1}^{n} a_i e_i,$$

where the e_i are pairwise disjoint idempotents of pA, and each $a_i \in A$. Note that in an f-ring an idempotent is necessarily positive. Then it is easy to verify that

$$x \vee 0 = \sum_{i=1}^{n} (a_i \vee 0)e_i,$$

which shows that $a \vee 0 \in pA$, and proves the proposition. ∎

We now have, directly from the above remarks and Proposition 6.2:

Corollary 7.5. *Suppose that A is an f-ring. Then HA is an ℓ-subring of QA.*

Remark 7.6. Suppose that A is projectable. Then, owing to Lemma 3.5(b) and [D95, Proposition 18.1], the set of prime convex ℓ-subgroups forms, relative to inclusion, a disjoint union of chains. In particular, so do the prime ℓ-ideals of A. Thus, each $\mathfrak{p} \in \mathrm{Min}(A)$ is contained in a unique maximal ℓ-ideal $\mathfrak{p}^*$, and the assignment $\nu_A : \mathfrak{p} \mapsto \mathfrak{p}^*$ from $\mathrm{Min}(A)$ to $\mathrm{m}(A)$, the space of all maximal ℓ-ideals of A with the hull-kernel topology, is a homeomorphism of compact Hausdorff spaces. (The continuity of ν_A is proved as in [BKW77, Lemma 10.2.3].)

Combining with Proposition 6.2, we obtain the following result. The homeomorphism in question is the composition

$$\mathrm{Min}(HA) \xrightarrow{\tau} \mathrm{Min}(pA) \xrightarrow{\nu_{pA}} \mathrm{m}(pA).$$

The reader should note that $\mathrm{Min}(HA) = \mathrm{m}(HA)$, since HA is von Neumann regular.

Proposition 7.7. *For any f-ring A, $\mathrm{m}(HA)$ and $\mathrm{m}(pA)$ are homeomorphic, and isomorphic to the Stone dual of $\mathcal{P}_\omega(A)$.*

For the next concept, the reader should refer to Proposition 2.6, 3.6 and [CM90].

Remark 7.8. Suppose that B is an f-ring, containing A as an ℓ-subring. We say that A is *rigid* in B if for each $b \in B$ there is an $a \in A$ such that $b^{\perp\perp} = a^{\perp\perp}$. Proposition 2.6 guarantees that A is rigid in B if and only if ε induces the isomorphism $d(\varepsilon)$ from $d\mathrm{Rad}(A)$ onto $d\mathrm{Rad}(B)$. The trace τ induces a homeomorphism from $\mathrm{Min}(B)$ onto $\mathrm{Min}(A)$.

Observe that if B lies between A and qA, then A is necessarily rigid in B.

We now have the following, which generalizes [RW99, 3.3]; we will remind the reader of this in the next section. The hypothesis in Proposition 7.9 is also cleaner than that of [St68, Lemma 3.3].

Recall that $E_{\mathcal{P}_\omega}(A)$ stands for the set of idempotents of the form $e(K)$, for all $K \in \mathcal{P}_\omega(A)$; per Corollary 5.4, this is also the boolean algebra of all idempotents of pA.

Proposition 7.9. *Suppose that A is an f-ring, and B is an ℓ-subring of qA, containing A. Then $HA = HB$.*

Proof. As already remarked, A is rigid in B, and this means that the trace τ induces an isomorphism of $\mathcal{P}_\omega(B)$ onto $\mathcal{P}_\omega(A)$. On the other hand, it should be clear that, for each $K \in \mathcal{P}_\omega(B)$, $e(K) \in pA$. This is to say, that $E_{\mathcal{P}_\omega}(A) = E_{\mathcal{P}_\omega}(B)$, and so pA and pB have the same idempotents. Each $b \in B$ is a fraction $b = \frac{a}{d}$, with $a, d \in A$, and this makes it clear that $B \leq HA$, thanks to Proposition 6.2. Thus, $HB \leq HA$, and the reverse inclusion is obvious. ∎

Remark 7.10. Let $h : A \longrightarrow B$ be a morphism in a category; if h is both monic and epic we shall call it an *epimorphic extension*. We point out, for later use, that, in the category $\mathfrak{R}$ of all commutative (and not necessarily semiprime) rings with identity, the extension $A \leq HA$ is epimorphic ([St68]).

It appears that Proposition 7.9 actually offers something new. Whereas qA always is an epimorphic extension of A, an intermediate subring need not be; consider, for example, $A = C(\alpha\mathbb{N})$, where $\alpha\mathbb{N}$ denotes the one-point compactification of discrete $\mathbb{N}$. Note that A is the ring of all convergent sequences real valued sequences. It is well known – see, for example, [HM93] – that $qA = QA = C(\mathbb{N})$, the ring of all real sequences. Now, the intermediate ring $C(\beta\mathbb{N})$ of all bounded sequences is not an epimorphic extension in $\mathfrak{Arf}$, the category of all archimedean f-rings with identity ([BH89, Chapter 8]), let alone in $\mathfrak{R}$.

Incidentally, one can verify directly that the criterion for epicity in [St68, Lemma 3.3], fails for the extension $C(\alpha\mathbb{N}) \leq C(\beta\mathbb{N})$.

In fact, pA is a proper subring of $C(\beta\mathbb{N})$; however, the same techniques from [BH89] show that $A \leq pA$ is not an epimorphic extension of A in $\mathfrak{Arf}$ either. Let us therefore emphasize: although $A \leq HA$ is an epimorphic extension, the inclusion $A \leq pA$ need not be epimorphic, even in $\mathfrak{Arf}$.

The rest of this section contains a study of the residue rings of prime d-ideals, in preparation for the representational issues of §8. We begin with a few remarks.

Remark 7.11. Let A be a semiprime f-ring. We note the following:

1. Every maximal ℓ-ideal which contains no regular elements is a d-ideal. This is easy to show directly; for the frame-theoretic counterpart, see [M06, 5.8].

2. Every d-ideal of A is an ℓ-ideal. This is most easily seen by noting that every d-ideal is an updirected join of annihilators. Thus, if $\mathfrak{p}$ is a prime d-ideal, then $A/\mathfrak{p}$ is a totally ordered integral domain.

Next, we have a proposition which links the residue ring of a prime d-ideal of A with the residue field of the maximal ideal of HA which traces upon it. For each

$\mathfrak{p} \in \mathrm{Spec}_d(A)$, recall that $\mu_d(\mathfrak{p})$ is the minimal prime ideal of pA which traces on $\mathfrak{p}$, and, in turn, there is a unique maximal ideal of $HA = qpA$, namely

$$\mathfrak{p}^* = \{\, a/d \in HA \,:\, a \in \mu_d(\mathfrak{p}),\ d \text{ regular in } pA \,\},$$

which traces on $\mu_d(\mathfrak{p})$.

Proposition 7.12. *Suppose that $\mathfrak{p} \in \mathrm{Spec}_d(A)$. Then*

(a) *The canonical homomorphism $A/\mathfrak{p} \longrightarrow pA/\mu_d(\mathfrak{p})$ is an order-isomorphism.*

(b) *The canonical embedding of $A/\mathfrak{p}$ into $HA/\mathfrak{p}^*$ embeds the former as an ordered subring of its field of fractions.*

Proof. (a) Pick $b \in pA$ and write it canonically as $b = b_1e_1 + \cdots + b_ne_n$, so that each e_i is an idempotent of pA, with $e_ie_j = 0$, for $i \neq j$, and each $b_i \in A$. Then, if $b \notin \mu_d(\mathfrak{p})$, exactly one $e_j \notin \mu_d(\mathfrak{p})$, so that

$$b + \mu_d(\mathfrak{p}) = b_je_j + \mu_d(\mathfrak{p}) = b_j + \mu_d(\mathfrak{p}).$$

This shows that the canonical homomorphism $A/\mathfrak{p} \longrightarrow pA/\mu_d(\mathfrak{p})$ is surjective. That it is one-to-one and preserves order is obvious.

(b) Since the embedding $A \leq HA$ is epimorphic – see 7.10 – the map $A \longrightarrow HA/\mathfrak{p}^*$, by $a \mapsto a + \mathfrak{p}^*$ is an epimorphism of $\mathfrak{R}$, and as a second factor of an epimorphism, the canonical $A/\mathfrak{p} \longrightarrow HA/\mathfrak{p}^*$ is also epic in $\mathfrak{R}$. We therefore have an epimorphic extension of an integral domain in a field. The latter must be the field of fractions of the former, by [Ro68, Theorem 2]. This completes the proof of the proposition, as the canonical embedding

$$A/\mathfrak{p} \cong pA/\mu_d(\mathfrak{p}) \longrightarrow HA/\mathfrak{p}^*$$

is order preserving. ■

In view of some of the work of [RW99], on realizing the regular hull as a ring of real valued functions, the following corollary is interesting. The first claim is obvious from Proposition 7.12. The second is a consequence of Hölder's Theorem ([D95, Corollary 24.17]).

Corollary 7.13. *Suppose that $\mathfrak{p} \in \mathrm{Spec}_d(A)$. Then $HA/\mathfrak{p}^*$ is a real field if and only if $A/\mathfrak{p}$ is archimedean. This happens, in turn, precisely when $\mathfrak{p}$ is a maximal convex ℓ-subgroup of A.*

8 Representing the Regular Hull

We discuss the various topologies on the structure spaces which are relevant to the projectable and regular hulls. Let us begin by recalling some general notions related to structure spaces. A throughout denotes a semiprime f-ring. The main theorem of this section is Theorem 8.9, characterizing the points of $\mathrm{m}(A)$ corresponding to real points of $\mathrm{Max}(HA)$.

Definition & Remarks 8.1. The hull-kernel topology on $\mathrm{Spec}(A) \equiv \mathrm{Spec}(\mathrm{Rad}(A))$ is generated by the basic open sets of the form

$$\mathrm{coz}(a) = \{\, \mathfrak{p} \in \mathrm{Spec}(A) \,:\, a \notin \mathfrak{p} \,\},$$

with $a \in A$. It is a compact, though (in general) not a Hausdorff space.

In the context of this article we have already considered several subsets of $\mathrm{Spec}(A)$, such as $\mathrm{m}(A)$ and $\mathrm{Spec}_d(A)$, as well as $\mathrm{Min}(A)$. It is well known that, under the subspace topology, $\mathrm{m}(A)$ is compact Hausdorff, for a semiprime f-ring A; $\mathrm{Spec}_d(A)$ is compact, but also not Hausdorff, in general. $\mathrm{Min}(A)$ is not generally compact, but it is Hausdorff and has a base of clopen sets. In [HdP83] $\mathrm{Max}_d(A)$, the space of maximal d-ideals, assumes some prominence; it is compact Hausdorff.

In view of Theorem 5.7 and Corollary 7.13, it would be interesting to examine the image of $\mathrm{Max}_d(A)$ under μ_d. We proceed to that. Let $\mathfrak{M} \equiv \mu_d(\mathrm{Max}_d(A))$, and $\mathfrak{M}_H$ denote the set of maximal ideals of HA which trace on an ideal in $\mathfrak{M}$.

We now record the following observation.

Proposition 8.2. *Assume that A is archimedean. Then $\bigcap \mathfrak{M} = \{0\}$; that is to say, $\mathfrak{M}$ is dense in $\mathrm{Min}(pA)$. Finally, $\mathfrak{M}_H$ is also dense in $\mathrm{Max}(HA)$.*

Proof. The last claim follows from the rest of the proposition, because $\mathrm{Max}(HA)$ and $\mathrm{Min}(pA)$ are homeomorphic under the trace map. From elementary considerations of the hull-kernel topology, the second assertion follows from the first. As to the first, note that since A is archimedean (see [HdP83, Proposition 2.3])

$$0 = \bigcap \mathrm{Max}_d(A) = A \cap \Big(\bigcap \mathfrak{M}\Big),$$

and since A is essential in pA, the claim follows. ∎

Using Corollary 7.13 and the preceding, we have the following characterization of when HA is representable faithfully by real valued functions. We leave the few details to the reader.

Corollary 8.3. *For an archimedean f-ring A the following are equivalent.*

(a) *There is an ℓ-embedding of HA into $\mathbb{R}^I$, the latter with pointwise ordering, for a suitable set I.*

(b) *There is a dense subfamily $\mathfrak{R}_H$ of $\mathfrak{M}_H$, such that $HA/\mathfrak{m}$ is a real field, for each $\mathfrak{m} \in \mathfrak{R}_H$.*

(c) *There is a dense subfamily $\mathfrak{R}$ of $\mathfrak{M}$, such that $pA/\mathfrak{p}$ is an archimedean domain, for each $\mathfrak{p} \in \mathfrak{R}$.*

(d) *There is a dense subfamily $\mathfrak{L}$ of $\mathrm{Max}_d(A)$ such that each $\mathfrak{p} \in \mathfrak{L}$ is a maximal convex ℓ-subgroup of A.*

(e) *There is a dense subfamily $\mathfrak{L}$ of $\mathrm{m}(A)$ such that each $\mathfrak{p} \in \mathfrak{L}$ is real and a d-ideal of A.*

Remark 8.4. Corollary 8.3 might, upon some reflection, seem too good to be true. The key to understanding it, especially regarding the equivalence of (d) and (e), is for the reader to consider that, in general, a maximal d-ideal need not be a maximal ℓ-ideal, and even when this is so, the ideal in question need not be a maximal convex ℓ-subgroup. We defer to Example 9.4.

When, and only when, the three coincide, is the ideal $\mathfrak{m} \in m(A)$ the trace of a real ideal of $H(A)$ – and per the upcoming formulation in Theorem 8.9 – an almost P-point of $m(A)$; see 8.7.

Let us look now at the consequence of Corollary 8.3 in the event that A is a uniformly complete f-algebra. Uniform completeness is used here with reference to a specific "regulator", namely, the identity of the algebra. Since the algebras under consideration are archimedean, uniform completeness in this sense implies the ostensibly stronger property of "relative" uniform completeness (with respect to all regulators). The interested reader is referred to [AH81].

Definition & Remarks 8.5. First, recall that a sequence f_n $(n = 1, 2, \ldots)$ in a real f-algebra A is *uniformly Cauchy* if for each $\varepsilon > 0$ there is a natural number k such that $|f_{n+k} - f_k| < \varepsilon$, for each $n \in \mathbb{N}$. f_n $(n = 1, 2, \ldots)$ *converges uniformly to* $f \in A$ if for each $\varepsilon > 0$ there is a natural number k such that $|f_n - f| < \varepsilon$, for each $n \geq k$. Finally, A is *uniformly complete* if every uniformly Cauchy sequence of A converges uniformly in A.

Recall ([HdP80a, Theorem 5.3]) that if A is uniformly complete and archimedean, and J is a d-ideal such that A/J is archimedean, then J is a σ*-ideal*; that is, J is closed under all existing suprema of countable sets.

It is convenient as well to be able to consider archimedean f-algebras in their Henriksen-Johnson Representation. We give a brief review of the pertinent ideas and facts.

Definition & Remarks 8.6. (a) Let A be an archimedean f-algebra; recall that $\mathrm{m}(A)$ stands for the space of its maximal ℓ-ideals. We wish to represent the elements of A on $\mathrm{m}(A)$ as "almost real valued" functions. We explain.

Suppose that X is a compact Hausdorff space. $\mathbb{R} \cup \{\pm\infty\}$ denotes the two-point compactification of the ordinary real numbers. $D(X)$ shall denote the set of all continuous functions $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, for which $f^{-1}(\mathbb{R})$ is dense. It is well known that $D(X)$ is a lattice under pointwise operations, though, in general, not a group or ring under the obvious pointwise operations. Nonetheless, we say that an f-algebra A which is a subset of $D(X)$ is *an f-algebra in $D(X)$* if for each $f, g \in$ there is a dense open subset U of X such that $f(x) + g(x) = (f + g)(x)$, for each $x \in U$, and also $f(x)g(x) = (fg)(x)$, for eahc $x \in U$.

We say that $A \subseteq D(X)$ *separates the points of* X if for each $x \neq y$ in X, there if an $f \in A$ such that $f(x) \neq f(y)$.

The Henriksen-Johnson Representation (see [HJ61]) can be formulated as follows:

> *Suppose that A is an archimedean f-algebra. Then there is an ℓ-isomorphism onto an f-algebra A' in $D(\mathrm{m}(A))$ which separates the points of* $\mathrm{m}(A)$.

(b) It is easily seen that Corollary 8.3 may be paraphrased as follows:

> *The conditions of the corollary are satisfied precisely when the Henriksen-Johnson Representation of HA on* $\mathrm{Max}(HA)$ *induces an ℓ-isomorphism of HA into $C(\mathfrak{R}_H)$, where $\mathfrak{R}_H$ denotes the set of real maximal ideals of* $\mathrm{Max}(HA)$.

We emphasize further that $\mathfrak{R}_H$ above is homeomorphic to the subspace $\mathfrak{L}$ of $\mathrm{Max}_d(A)$, consisting of all the d-ideals which are also maximal convex ℓ-subgroups, with respect to the patch topology on $\mathfrak{L}$.

Definition 8.7. Let X be a topological space; $p \in X$ is an *almost P-point* if for each $f \in C(X)$, such that $f(p) = 0$, there is a nonvoid open set V on which f vanishes. Isolated points are almost P-points; the point at infinity in the one-point compactification of an uncountable discrete set D is the simplest example of a nonisolated almost P-point. X is an *almost P-space* if every point is an almost P-point.

For the basics on almost P-points and almost P-spaces the reader is referred to [Le77].

For the rest of this section we shall identify an archimedean f-algebra with its image under the Henriksen-Johnson Representation. We are now ready for the main result of this section. One part of the converse of Theorem 8.9 does not require the assumption of uniform completeness in the f-algebra. We record that part separately in a lemma.

Lemma 8.8. *Suppose that A is an archimedean f-algebra. If $x \in \mathrm{m}(A)$ is an almost P-point, then $\mathfrak{m}_x = \{\, f \in A \,:\, f(x) = 0 \,\}$ is a real ideal.*

Proof. Suppose that $g(x) = \infty$, for some $g \in A$. Without loss of generality, $g \geq 1$. Now g^{-1} (which lies in $C(\mathrm{m}(A))$) vanishes at x, but has a nowhere dense zeroset. This contradicts the assumption that x is an almost P-point. ∎

Theorem 8.9. *Suppose that A is a uniformly complete archimedean f-algebra. Let $\mathfrak{m} \in \mathrm{m}(A)$ be the trace of a real maximal ideal of HA. Then $\mathfrak{m}$ is an almost P-point of $\mathrm{m}(A)$. Conversely, if x is an almost P-point of $\mathrm{m}(A)$, then $\mathfrak{m}_x$ is a real ideal as well as a d-ideal, and therefore it is a real ideal in $\mathrm{Max}_d(A)$. Consequently, $\mathfrak{m}_x$ is the trace of a real maximal ideal of HA.*

Proof. Thinking of the elements of A as functions on $\mathrm{m}(A)$, we shall write $f(\mathfrak{p})$ for the image under the Henriksen-Johnson Representation.

First assume that $\mathfrak{m} \in \mathrm{m}(A)$ is the trace of a real maximal ideal of HA. Now, according to Corollary 7.13, $\mathfrak{m}$ is a maximal convex ℓ-subgroup as well as a d-ideal. In particular, it is a point in $\mathrm{m}(A)$. Now suppose, by way of contradiction, that some $f \in C(\mathrm{m}(A))$ vanishes at $\mathfrak{m}$, yet f is regular. On account of the Stone-Weierstrass Theorem, $C(\mathrm{m}(A)) \cong A^*$, where A^* stands for the subring of A consisting of the functions of A bounded by a constant multiple of the identity. Thus, we may consider f as an element of A.

We claim that

$$1 = \bigvee_{n=1}^{\infty} (nf \wedge 1).$$

For suppose that some $g \in A$ exceeds each $nf \wedge 1$, with $g < 1$. Then for each point $\mathfrak{p} \in \mathrm{coz}(1-g)$, we must have $nf(\mathfrak{p}) \leq g(\mathfrak{p})$, which implies that $f(\mathfrak{p}) = 0$. Since $1-g > 0$, this contradicts the assumption about f. The claim is therefore established.

Next, as observed in 8.5, $\mathfrak{m}$ is a σ-ideal, and since each $nf \wedge 1$ vanishes at $\mathfrak{m}$, so does 1, which is absurd. This contradiction to the assumptions about f shows that $\mathfrak{m}$ must be an almost P-point of $\mathrm{m}(A)$.

Conversely, suppose that $x \in \mathrm{m}(A)$ is an almost P-point. To simplify the notation in this argument, let us use $X = \mathrm{m}(A)$. We also employ the notation of Lemma 8.8. By that lemma, $f(x) \in \mathbb{R}$, for each $f \in A$. If x is isolated there is nothing to prove, as $\mathfrak{m}_x$ is a minimal prime ideal. We may therefore assume that x is not isolated.

Next, by way of contradiction, suppose that there is an $a > 0$ in $\mathfrak{m}_x$ for which $a^{\perp\perp} \not\subseteq \mathfrak{m}_x$. Observe first of all that $x \in \mathrm{cl}_X \mathrm{int}_X(Z(a) \setminus \{x\})$; for if not, there is a bounded $g \in A^+$ such that $g(x) = 0$ and g restricted to $\mathrm{int}_X(Z(a) \setminus \{x\})$ is identically 1. But then it is easily checked that $a + g$ is regular, while $a + g$ vanishes at x, contradicting that x is an almost P-point. We now pick $0 < b \in a^{\perp\perp} \setminus \mathfrak{m}_x$. As $b(x) > 0$ there is a point $y \in \mathrm{int}_X(Z(a) \setminus \{x\})$ with $b(y) > 0$. So we may also find $c \in A^+$ such that $y \in \mathrm{coz}(c) \subseteq Z(a) \setminus \{x\}$, which implies, since x is not isolated, that $ac = 0$, whence also $bc = 0$, a contradiction, since both b and c are nonzero at y. This contradiction shows that $\mathfrak{m}_x$ is a d-ideal, and the theorem is proved, as the remaining claims are, by now, obvious. ∎

Corollary 8.3(e) and Theorem 8.9 now gives us the following tidy characterization.

Corollary 8.10. *Suppose that A is a uniformly complete archimedean f-algebra. Then HA can be represented as a ring of real valued functions if and only if* $\mathrm{m}(A)$ *has a dense subset of almost P-points.*

Remark 8.11. Suppose that X is a Tychonoff space. If $p \in X$ is an almost P-point of X, then p is an almost P-point of any dense subspace that contains p. This is easy to see, when one realizes the (well known) fact that $p \in X$ is an almost P-point if and only if each G_δ-set that contains it has nonempty interior. Corollary 8.10 implies that if A is uniformly complete and archimedean and HA admits a Henriksen-Johnson Representation by real valued functions, then $\mathrm{m}(A)$ contains a dense almost P-space.

9 Examples.

The first two examples demonstrate that in the passage from A to pA, for a prime d-ideal $\mathfrak{p}$ of A, $\mu_d(\mathfrak{p})$ may properly contain $d(\varepsilon)(\mathfrak{p})$, while the latter may fail to be a prime ideal of pA. It also shows that the trace map, on all the d-ideals of pA is not one-to-one, and, restricted to the minimal prime ideals of pA, may have a nonminimal trace.

Further, $\mathrm{Min}(pA)$ is not necessarily the Stone-Čech compactification of $\mathrm{Min}(A)$. There is an additional note of caution for the reader in the second example.

Example 9.1. We consider the ring $R = \mathbb{R}[[T]]$, of all formal power series in one variable over the reals. Next let B denote the direct product of countably many copies of R. This is a semiprime ring, the identity of which is (1,1,...). Finally, A is the subring of all tuples $(f_1, f_2, \ldots)$ in B for which the constant terms $f_i(0)$ all agree, and also f_i is a constant, for all but finitely many i.

Now $\mathrm{Spec}_d(A)$ consists of the minimal prime ideals

$$\mathfrak{p}_n \equiv \{\, (f_1, f_2, \ldots) \in A \,:\, f_n \equiv 0 \,\},$$

for each $n \in \mathbb{N}$, and the lone maximal d-ideal

$$\mathfrak{m} \equiv \{\, (f_1, f_2, \ldots) \in A \,:\, f_n(0) = 0, \forall\, n \,\}.$$

Incidentally, $\mathrm{Min}(A)$ is a countable discrete space.

The reader will easily verify that pA is the subring of B generated by the direct sum of the copies of R and the identity of B. Putting it differently, $f = (f_1, f_2, \ldots) \in pA$ if and only if f_n is constant, for all but finitely many n. The reader should not have any trouble checking all of the following features:

(i) $f \in d(\varepsilon)(\mathfrak{p}_n)$ if and only if $f_n \equiv 0$, and $f_m(0)$ is eventually zero. This is not a prime ideal, as we have $ab = 0$ with $a, b \notin d(\varepsilon)(\mathfrak{p}_n)$, with $a = (a_1, a_2, \ldots)$ defined by $a_n = 1$ and $a_m = 0$, for each $m \neq n$, and $b = (b_1, b_2, \ldots)$ defined by $b_n = 0$ and $b_m = 1$, for each $m \neq n$.

(ii) By contrast, $f \in \mu_d(\mathfrak{p}_n)$ if and only if $f_n \equiv 0$.

(iii) $d(\varepsilon)(\mathfrak{p}_n) = \mu_d(\mathfrak{p}_n) \cap \mu_d(\mathfrak{m})$, where $\mu_d(\mathfrak{m})$ is the ideal consisting of all f such that f_n is eventually the zero series. Note that $\mathfrak{m} = \tau(\mu_d(\mathfrak{m}))$, that is, a nonminimal prime ideal of A which is the trace of a minimal prime.

(iv) $\mathrm{Min}(pA)$ is homeomorphic to the one-point compactification of the discrete countable space.

The reader will easily find ideals of A which have no regular elements, yet are not d-ideals.

Example 9.2. Let D be an uncountable, discrete space, and consider αD, the one-point compactification of D, and $A = C(\alpha D)$. Denote the point at infinity by ∞.

(a) In [RW99, Example 6.1] the authors show that HA is the ring $C(\lambda D)$, where λD stands for the space obtained by adjoining a point λ "at infinity" to D, so that its neighborhoods are the subsets which have countable complement in D. Now $\mathrm{m}(C(\lambda D)) = \beta\lambda D$, and λ is a *P-point* in $\beta\lambda D$; this means that any countable intersection of neighborhoods of λ is a neighborhood of λ. Thus, by [GJ76, 4L], it follows that M_λ is both a maximal and minimal prime ideal. $M_\lambda \cap pA$ is still minimal in pA.

On the other hand, $M_\lambda \cap A = M_\infty$, the maximal ideal of A of all functions which vanish at ∞. M_∞ is maximal in A, but not minimal.

(b) If $x \in \beta\lambda D \setminus \{\lambda\}$ then, either $x \in D$ and isolated, in which case M_x traces to the ideal of functions in A which vanish at x, or else x corresponds to a free ultrafilter $\mathfrak{F}_x$ on D which contains some countably infinite subset of D. Then M_x traces to the ideal P_x of A, contained in M_∞, consisting of all functions which vanish on some set of $\mathfrak{F}_x$. This is a minimal prime ideal of A. Thus, $\mathrm{Min}(A)$ is homeomorphic to $\beta\lambda D \setminus \{\lambda\}$, and therefore not C^*-embedded in $\mathrm{Min}(HA)$. Putting it differently: $\mathrm{Min}(HA)$ is not the Stone-Čech compactification of $\mathrm{Min}(A)$.

(c) This is by way of preface to the following observation; it stands in contrast to what will be said below about maximal ideals. For each point x corresponding to an ultrafilter $\mathfrak{F}_x$ (as in (b)), P_x is the trace of a prime ideal of $HA = C(\lambda D)$, although the ideal $\bar{P}_x$ generated in HA (or pA) by P_x is not prime. We let the reader verify this.

We mention this to disabuse the reader of the following argument. Note that $P_x = \bar{P}_x \cap A$, and the natural homomorphism $A \longrightarrow HA/\bar{P}_x$ is an epimorphism in $\mathfrak{R}$, as it is the composition of two such epimorphisms. Thus, as a second factor of an epimorphism, the natural embedding $A/P_x \longrightarrow HA/\bar{P}_x$ too is an epimorphism of $\mathfrak{R}$. However, it cannot be essential, for if so, then (since A/P_x is an integral domain) we would be able to conclude that $HA/\bar{P}_x$ (being von Neumann regular) is the field of fractions of A/P_x. And so $\bar{P}_x$ would be not only prime but maximal! This is absurd.

For complications of the type suggested by this example, the reader is referred to [Ro68], Proposition 6 and its corollary.

Remark 9.3. Example 9.2 also shows that if $\mathfrak{m}$ is a maximal d-ideal of A, then $\mathfrak{m}'$, the ideal it generates in pA, need not be maximal nor a d-ideal.

Note that $pA \leq C(\beta\lambda D)$. In fact, pA is the subring of all $f \in C(\beta\lambda D)$ such that f is constant off a countable subset S of D, while $f|_S$ is a finite disjoint supremum of convergent sequences.

Next consider $\mathfrak{m} = M_\infty$. It consists of the functions f on D with countable cozeroset $d_1, d_2, \ldots$, such that $\lim_n f(d_n) = 0$. It is easily seen that $\mathfrak{m}$ is an ideal of pA as well; that is, $\mathfrak{m} = \mathfrak{m}'$, and it is not a maximal ℓ-ideal of pA, as $M_p \cap pA$ contains it properly. Observe as well that $\mathfrak{m}$ is not a d-ideal of pA. Indeed, $d(\varepsilon)(\mathfrak{m}) = M_p \cap pA$.

Finally, here are the examples announced in 8.4.

Example 9.4. It is well known that if the ring A is complemented, then every prime d-ideal is minimal. This is shown in frame-theoretic generality in [MZ03]. This observation applies to both examples exhibited here.

(i) Consider $C(\alpha\mathbb{N})$, and refer to 7.10. The maximal ℓ-ideals are the maximal ideals, and also coincide with the maximal convex ℓ-subgroups. Since $C(\alpha\mathbb{N})$ is complemented, each prime d-ideal is minimal, and therefore the only prime d-ideals which are maximal ideals are the ideals $\mathfrak{m}_p$ of the functions vanishing at an isolated point $p \in \alpha\mathbb{N}$.

   Note as well that $H(C(\alpha\mathbb{N})) = C(\mathbb{N})$, and in terms of the conditions of Theorem 8.9, the reader should also observe that the lone nonmaximal ideal is not a d-ideal, and not an almost P-point of $\mathrm{m}(C(\alpha\mathbb{N}))$.

(ii) Now consider $C(\mathbb{R})$, which is also complemented. None of the minimal prime ideals of this ring are maximal. This implies that $H(C(\mathbb{R}))$ has no real ideals whatsoever. (The knowledgeable reader will realize that $H(C(\mathbb{R})) = q(C(\mathbb{R})) = Q(C(\mathbb{R}))$ – see [HM93], and that the maximal ideal space of this ring is the absolute of the Cantor set; the latter has no P-points, which implies that there are no real ideals.)

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