Free Meets in Algebraic Frames

J. Martínez & W. Wm. McGovern

Abstract. In J. T. Wilson's doctoral dissertation, the author takes up the subject of *free meets*, that is, subsets S of a frame L whose meet $\bigwedge S$ is preserved by a designated class of frame homomorphisms out of L. Wilson proves that all subsets of L have free meets if and only if the dual frame law holds in L and the assembly $\mathbb{N}L$ is a boolean frame.

In this paper it is shown that for algebraic frames L with the FIP and disjointification, the following are equivalent: (a) L satisfies the dual frame law and $\mathrm{Spec}(L)$ the descending chain condition; (b) all subsets of L have free meets; (c) all subsets of L have free meets with respect to all coherent surjective frame maps. If L has these properties, then the assembly is, in fact, atomic.

1 Introduction

The assembly of a frame L is the frame $\mathbb{N}L$ of nuclei defined on L, ordered pointwise. There is a natural embedding of L in its assembly; it assigns $x \in L$ to the closed nucleus c_x determined by x. The assembly has been studied by several authors, notably in [Wi94], and most recently in [SS04, Si06, MMZ08]. Some aspects of the way L lies embedded in its assembly are well understood.

In this paper the free meets of a frame are considered as a relative feature: freeness is to be construed *vis-à-vis* a designated class of morphisms out of the frame of discourse. In his thesis ([Wi94]) J. T. Wilson proves a remarkable and intriguing theorem: *A frame L has free meets – relative to all frame maps out of L – if and only if (a) the dual frame law is satisfied by L and (b) $\mathbb{N}L$ is boolean.* Using Conrad's Theorem on finite-valued ℓ-groups – adapted to algebraic frames with disjointification – it is shown here (Theorem 3.9) that L has free meets for all maps out of L precisely when L satisfies the dual frame law and $\mathrm{Spec}(L)$ the descending chain condition. This is established in three stages:

- (Theorem 3.3) First, all meets of L are free with respect to all coherent, surjective frame maps from L if and only if L satisfies the dual frame law and $\mathrm{Spec}(L)$ the descending chain condition. (A frame homomorphism that carries compact elements to compact elements is said to be *coherent*.)

- (Theorem 3.5) A characterization of the atoms in $\mathbb{N}L$ produces a sharpened consequence of Wilson's result: *When all meets in L are free, then $\mathbb{N}L$ is an atomic boolean algebra.*
- Further consideration of the techniques used in proving Theorems 3.3 and 3.5 lead to the aforementined result.

We assume general familiarity by the reader with elementary frame theory. We refer the reader to [J82] and [PT01] for any unexplained concepts and terms. However, we do not follow the topologist's conceit, under which one passes from frames to locales and reverses all arrows. Here frames retain their identity, as do the frame maps between them.

Let L be a frame. A function $p : L \longrightarrow L$ is a *prenucleus* if it is inflationary, order preserving and satisfies

$$p(x \wedge y) \geq p(x) \wedge y.$$

The set $\mathbb{P}L$ of prenuclei is closed under composition of functions, arbitrary joins, and finite meets, so it is a subframe of the frame of all functions, with respect to pointwise operations. An idempotent – $p(p(x)) = p(x)$ – prenucleus is a *nucleus.* For each prenucleus p, there is a nucleus $\bar{p}$ with the same fixed points; $\bar{p}$ is the smallest idempotent prenucleus above p. In fact, $\bar{p}$ may be obtained from p by transfinite iteraton of composites of p, but that is not relevant to our purposes.

What is needed, crucially, in the proof of Theorem 3.9 is this. A moment's reflection will make the claim transparent.

Lemma 1.1. *Suppose L is a frame and p is a prenucleus on L. Then $\bar{p} = 1$ – that is $\bar{p}(x) = 1$, for each $x \in L$ – if and only if $p(x) > x$ for every $x < 1$.*

The map $p \mapsto \bar{p}$ is itself a prenucleus on $\mathbb{P}L$. Therefore, the set $\mathbb{N}L$ of all nuclei is a frame. However, joins in the frame of nuclei cannot generally be computed by taking pointwise joins of functions. The frame $\mathbb{N}L$ is called the *assembly* of L. To understand the assembly, it is necessary to consider some particular nuclei.

Definition & Remarks 1.2. Throughout these remarks, L denotes a frame. For the record, the bottom 0 in $\mathbb{N}L$ is the identity function, whereas the top 1 is the nucleus which inflates L to 1.

Recall first that $x \to y$ denotes the *Heyting operation*, defined by

$$z \leq x \to y \iff x \wedge z \leq y.$$

Given $a \in L$, the functions

$$c_a(x) = a \vee x \quad \text{and} \quad o_a(x) = a \to x$$

are nuclei. These nuclei satisfy ([J82, II, Lemma 2.6])

$$c_a \overset{\mathbb{N}L}{\vee} o_a = 1 \quad \text{and} \quad c_a \wedge o_a = 0;$$

that is, they are complementary nuclei. Motivated by topology, c_a is referred to as *the closed quotient of L by a* and o_a as *the open quotient of L by a.* Note that

- the map $c : L \longrightarrow \mathbb{N}L$ by $a \mapsto c_a$ is a frame embedding, while
- $o : L \longrightarrow \mathbb{N}L$ by $a \mapsto o_a$ is one-to-one and satisfies

$$o_{(a\wedge b)} = o_a \vee o_b, \quad \text{and} \quad o_{(\bigvee S)} = \bigwedge \Big\{ o_s \,:\, s \in S \Big\}.$$

(See [Wi94], Proposition 15.1,(e) and (f).)

The assembly is generated by the complemented open and closed nuclei.

Proposition 1.3. ([J82, II, Proposition 2.7]) *Let L be any frame. Then $\mathbb{N}L$ is generated (by suprema) of nuclei of the form $c_a \wedge o_b$.*

We are concerned in this article with algebraic frames: that is, with frames in which every element is a join of compact ones. We use $\mathfrak{k}(L)$ to denote the join-subsemilattice of all compact elements of the frame L. We say that L *has the finite intersection property (FIP)* if the meet of two compact elements of L is compact.

Finally, we recall (from [MMZ08]) the notion of a *smooth* filter: a filter S of the form $S = j^{-1}\{1\}$, for some nucleus j on L. In [NR88] smooth filters are said to be *closed*, while Simmons calls them *admissible* in [Si06].

2 Free Meets.

In [Wi94], Wilson defines a subset S of a frame L to *have a free meet* if for each frame map $h : L \longrightarrow M$, $h(\bigwedge S) = \bigwedge h(S)$. He gives several characterizations in §25. We highlight [Wi94, Theorem 25.5]: *A filter S is smooth if and only if S is closed under meets of subsets $T \subseteq S$ having free meets.* Further, the characterization of when every set in L has free meets, encapsuled by Proposition 2.3(a) below, is complete and elegant enough, albeit not very informative, as it is not transparent when the assembly of a frame is boolean. Our goal here is to shed some light on the structure of this boolean assembly. In this section the groundwork is lain; the structure theorems are in the next.

An additional distinction between this section and the next lies in the set-theoretic assumptions: we adhere to Zermelo-Fraenkel plus Countable Dependent Choice now, but invoke full Choice ahead.

We begin by sharpening the definitions.

Definition 2.1. It is assumed that L is a frame, $S \subseteq L$, and $\mathcal{H}$ is a class of frame homomorphisms whose domain is L.

We say that S has *$\mathcal{H}$-free meets* (or *free meets with respect to $\mathcal{H}$*) if $h(\bigwedge S) = \bigwedge h(S)$, for each $h \in \mathcal{H}$. If $\mathcal{H}$ is the class of all frame maps out of L, we drop mention of $\mathcal{H}$, and adopt Wilson's usage.

Following Wilson then, L has *free meets* if every subset of L has free meets.

Let us put together some observations regarding free meets in language better suited to matters involving the assembly. Most of the assertions are fairly obvious, or else they appear in either [C65] or [Wi94].

The reader is reminded of some fairly standard terminology pertaining to lattices – *distributive* lattices, in this context.

Remark 2.2. In a frame L

- $m \in L$ is *meet-irreducible* if $m = \bigwedge S$, with $S \subseteq L$, implies that $m \in S$. Evidently, every meet-irreducible element is prime. Note that, to assume, as in Proposition 2.3(c) below, that each $x \in L$ is the meet of meet-irreducible elements makes, is to make L spatial.
- The dual of a prime element is a *finitely join-irreducible* element: $a = x \vee y$ implies that either $a = x$ or $a = y$. Likewise, the dual of a meet-irreducible element is a *join-irreducible* element. Observe for later use that a compact finitely join-irreducible element is necessarily join-irreducible.
- We say that $c \in \mathfrak{k}(L)$ is *indecomposable* if $c = a \vee b$, with $a \wedge b = 0$, implies that a or b is c.

Recall that L is *joinfit* if for each $0 < a \in L$, there is a $z < 1$ in L such that $a \vee z = 1$. This concept, obviously a kin to the better-known fitness conditions, was introduced in [M08b]. It is the frame-theoretic counterpart of semisimplicity in commutative rings, and of archimedeaneity in lattice-ordered groups.

Proposition 2.3. *Suppose that L is a frame.*

(a) *L has free meets if and only if the following two conditions hold:*

  (i) *L satisfies the dual frame law;*

  (ii) *$\mathbb{N}L$ is a boolean frame.*

  ([Wi94, Theorem 26.1])

(b) *If L has free meets, then each smooth filter S of L is principal – i.e., $S = \uparrow a$, for some $a \in L$.*

(c) *Suppose each $x \in L$ is the meet of meet-irreducible elements. Then L satisfies the dual frame law if and only if it is completely distributive; that is, for all $x_{ij} \in L$,*

$$\bigwedge_{i\in I} \bigvee_{j\in J} x_{ij} = \bigvee_{f\in J^I} \bigwedge_{i\in I} x_{if(i)} \quad ([C65]).$$

(d) *If L has free meets, then for each $a > 0$ there is a $b \in L$ such that $a \vee b = 1$ and*

$$(a \wedge b) \vee z = 1 \implies z = 1.$$

In particular, if L is also joinfit, then it is a boolean frame.

(e) *If L has free meets with respect to all frame surjections of L, then every prime is meet-irreducible. Thus,* Spec(L) *satisfies the descending chain condition.*

Proof. (b) According to [Wi94, Theorem 25.5], a smooth filter S is closed under every infimum of a subset having free meets. In particular, $\bigwedge S$ is in S.

(d) By (b), the filter $\Gamma_a \equiv \{\, x \in L : x \vee a = 1 \,\}$ has a least element b. Now, if $(a \wedge b) \vee z = 1$, then, simultaneously, $b \leq z$ and $b \vee z = 1$, whence $z = 1$, as claimed.

(e) Suppose $p \in L$ is prime. Let j_p be the associated character: $j_p(x) = 0$, if $x \leq p$, while $j_p(x) = 1$, otherwise. Now, as j_p preserves all meets, $\bigwedge S = p$ implies that $\bigwedge j_p(S) = 0$; this can be only if $p \in S$.

Since the meet of a chain of primes is prime, the remaining assertion is clear. ■

Applied to algebraic frames, the assumption of the dual frame law harkens back to Conrad's famous theorem, characterizing finite-valued lattice-ordered groups – see [C65], or [D95, §46]. While this is not the place for a full-blown digression on Conrad's theorem, it will be relevant to dig a little deeper with the assumption of the dual frame law. Armed with the Zermelo-Fraenkel axioms (ZF), plus the Axiom of Countably Dependent Choice (ZF+ACDC), one has the following lemma.

The proofs of the next two lemmas appear in [M07].

As observed in [M07], the proof of Lemma 2.4 only uses the distributivity of finite joins over countable meets, whereas Lemma 2.5 seems to require the full force of the dual frame law.

Lemma 2.4. (ZF+ACDC) *Suppose that L is an algebraic frame satisfying the dual frame law. Then every compact element of L may be expressed, uniquely, as a finite supremum of pairwise disjoint indecomposable compact elements.*

Recall that an algebraic frame L has *disjointification* if for each pair $a, b \in \mathfrak{k}(L)$, there exist disjoint c, d, both compact, such that $c \leq a$, $d \leq b$, and

$$a \vee b = c \vee b = a \vee d.$$

Equivalently, L has disjointification if and only if each $\downarrow a$ (with a compact) is normal.

Banaschewski calls this property "coherent normality", while in [ST93] it is called "relative normality". In spite of the obvious connection to normality, "disjointification" seems to better cut to the quick.

Lemma 2.5. (ZF) *Suppose that L is an algebraic frame with disjointification, satisfying the dual frame law. Then every indecomposable compact element is join-irreducible.*

One then has the following version of the result we shall refer to as Conrad's Theorem in these pages, and this account of it is valid in (ZF+ACDC).

Theorem 2.6. (ZF+ACDC) *For an algebraic frame L with disjointification, in which each element is the infimum of meet-irreducible elements, the following are equivalent:*

(a) *L is completely distributive.*

(b) *The dual frame law holds in L.*

(c) *Each $a \in \mathfrak{k}(L)$ can be decomposed, uniquely, as a finite supremum of pairwise disjoint join-irreducible elements.*

Proof. As was observed in Proposition 2.3(c), conditions (a) and (b) were shown to be equivalent in [C65], in ZF. Lemmas 2.4 and 2.5 establish that (b) implies (c). The relevant argument in the proof of [M72, Theorem 3.1] shows that (c) implies (a), and does not involve Choice issues. ∎

Remark 2.7. Notably absent in the formulation of Conrad's Theorem is any mention of the FIP. In fact, the FIP is a consequence of the theorem ([M08a]).

Remark 2.8. Assume the hypotheses of Conrad's Theorem.

As already noted in 2.2, a finitely join-irreducible compact element is join-irreducible. Following the usage in the theory of lattice-ordered groups, we shall refer to a compact join-irreducible element as being *special*.

For each special element c, let $m(c)$ denote the supremum of all elements *strictly* less than c. Then $m(c) < c$ and clearly the largest element under c. Now consider

$$v(c) = c \to m(c).$$

It is a routine matter to check that

- $v(c)$ is meet-irreducible and maximal with respect to $c \not\leq v(c)$;
- $v(c)$ is the unique meet-irreducible element which is maximal with respect to $c \not\leq y$.
- For any two special elements $a, b \in L$,

$$a < b \iff m(a) < m(b) \iff v(a) < v(b).$$

Let $\mathcal{S}(L)$ stand for the set of special elements of L.

The following corollary is immediate from all that has gone before in this section.

Corollary 2.9. *Suppose that L is an algebraic frame with disjointification, in which each element is the infimum of meet-irreducible elements.*

(a) *If, in addition, L has free meets, then $\mathcal{S}(L)$ satisfies the descending chain condition.*

(b) *If the dual frame law holds and $a, b \in \mathcal{S}(L)$, then either a and b are comparable or else disjoint.*

(c) *If (a) and (b) hold and $S \subseteq \mathcal{S}(L)$ then $\bigwedge S = 0$, unless S is a chain, in which case $\bigwedge S$ is its least element.*

3 Algebraic Frames with Free Meets.

In this section it is assumed that L is an algebraic frame with the FIP and having disjointification. We consider, first, free meets relative to all coherent, surjective frame maps out of L. Let us refer to such frames as those *having free meets under coherent quotients.* Our objective is Theorem 3.9, with major preliminary steps in Theorems 3.3 and 3.5.

We note, with emphasis, that in this section Zorn's Lemma is assumed, throughout. In particular, in every algebraic frame with the FIP, each $x \in L$ is an infimum of meet-irreducible elements. Indeed, $m \in L$ is meet-irreducible if and only it is maximal with respect to $a \not\leq m$, for a suitable compact $a \in L$. We say that m is a *value* of a. As already noted in 2.8, a special element has only one value. If m is the value of a special element, then m is also said to be *special.*

We use the terms *value* and *meet-irreducible element* interchangeably. The set of all vaues of L is denoted $\mathcal{V}(L)$.

With the assumption of Zorn's Lemma, one is able to add two more conditions to Conrad's Theorem. We phrase this as follows, referring the reader to [M72] or [ST93] for details. It is on account of the equivalence of Proposition 3.1(b) with the ones in Theorem 2.6, that frames with this property will be referred to as *finite-valued frames.*

Proposition 3.1. *For a frame L each of the following is equivalent to each of the conditions in Theorem 2.6.*

(a) *Each meet-irreducible element of L is special.*

(b) *Each compact element of L has at most finitely many values.*

Proposition 3.1 simplifies the "implementation" of Theorem 2.6 somewhat. The following lemma illustrates this.

Lemma 3.2. *Assume that $h : L \longrightarrow M$ is a coherent frame surjection, and that L is an algebraic frame with the FIP.*

(a) *If L has disjointification, then so does M.*

(b) *If, in addition, L is a dual frame, then so is M.*

Proof. (a) This is a consequence of a well known result of Monteiro ([Mo54]), stating that an algebraic frame L with the FIP has disjointification if and only if in Spec(L) every $\uparrow p$ is a chain.

(b) Given (a), we apply Proposition 3.1(b), as follows. First, every compact element of M is the image under h of a compact element of L. Then one simply observes that Proposition 3.1(b) is preserved by all coherent frame surjections. ■

We are now ready to prove the first structure theorem of this section; it is the first major step toward Theorerm 3.9.

Theorem 3.3. *Suppose that L is an algebraic frame with the FIP and disjointification. Then L has free meets under coherent quotients if and only if the dual frame law holds in L and $\mathcal{S}(L)$ satisfies the descending chain condition.*

Proof. (Necessity) Each c_x coherent, and the character used in the proof of Proposition 2.3(c) is coherent.

(Sufficiency) Let $S \subseteq L$, $h : L \longrightarrow M$ be a coherent frame surjection, and consider $\bigwedge S$. Express each member of S as a join of special elements; rewrite $\bigwedge S = \wedge_{j\in J} \vee_{i\in I} x_{ij}$ with each x_{ij} special. Then, since L is completely distributive we have

$$h\Big(\bigwedge\nolimits_{j\in J} \bigvee\nolimits_{i\in I} x_{ij}\Big) = \bigvee\nolimits_{f\in I^J} h\Big(\bigwedge\nolimits_{j\in J} x_{f(j)j}\Big). \tag{3.3.1}$$

Applying (c) of Corollary 2.9, each $\wedge_{j\in J} x_{f(j)j} = 0$, unless the $x_{f(j)j}$ forms a chain, in which case it is well-ordered and has a least element m_f. Thus, we note that $h(\wedge_{j\in J} x_{f(j)j}) = 0$, in the former case, and $h(m_f)$, in the latter.

In either of the above cases we have

$$h\Big(\bigwedge\nolimits_{j\in J} x_{f(j)j}\Big) = \bigwedge\nolimits_{j\in J} h(x_{f(j)j}). \tag{3.3.2}$$

However, by Lemma 3.2(b), M is completely distributive, which implies (putting together (3.3.1) and (3.3.2)),

$$h\Big(\bigwedge\nolimits_{j\in J} \bigvee\nolimits_{i\in I} x_{ij}\Big) = \bigvee\nolimits_{f\in I^J} \bigwedge\nolimits_{j\in J} h(x_{f(j)j}) = \bigwedge\nolimits_{j\in J} h\Big(\bigvee\nolimits_{i\in I} x_{ij}\Big). \tag{3.3.3}$$

The preceding identity demonstrates that the original S has a free meet under h. ■

Next, we identify the atoms in the assembly of L. It should be clear that if $j \in \mathbb{N}L$ is an atom then it is of the form $c_a \wedge o_b$. Moreover, $c_a \wedge o_b > 0$ precisely when $a \nleq b$. Therefore, without loss of generality, a is compact and b is meet-irreducible. If, in addition, L has the dual frame law, then from all of the foregoing, a may be assumed to be special with b the lone value of a.

We now demonstrate that, when L is finite-valued, those are indeed the atoms of the assembly. Recall that if $0 < a \in L$ is special, then $v(a) = a \to m(a)$ is the unique value of a.

For the lemma and theorem that follow, we assume that L is a finite-valued frame, subject as well to the standing assumptions throughout this section.

Lemma 3.4. *Let $a > 0$ be special; put $v = v(a)$, and $j_{a,v} = c_a \wedge o_v$; that is,*

$$j_{a,v}(x) = (a \vee x) \wedge (v \to x),$$

for each $x \in L$. Then

(a) *$j_{a,v}$ is an atom in $\mathbb{N}L$, and each atom is of this form.*

(b) *Each nontrivial nucleus exceeds some $j_{a,v}$.*

(c) *Each atom of $\mathbb{N}L$ is complemented.*

Proof. First, observe that if j is a nontrivial nucleus, there exist $c, d \in L$ such that $j \geq c_c \wedge o_d > 0$, and as was explained in the preamble to this lemma, $j \geq j_{a,v} > 0$, for some special a. Thus, (b) of the lemma, as well as the second claim of (a), follow from the first part of (a). As for (c), each $j_{a,v}$ is a meet of two complemented elements, and therefore also complemented.

To show $j_{a,v}$ is an atom, it is enough to establish that, f or any two distinct special elements a and b, $j_{a,v} \wedge j_{b,w} = 0$ (with $w = v(b)$). This is obvious if $a \wedge b = 0$. Otherwise, by Corollary 2.9(b), we may suppose that $a > b$. This implies that $v > w$ and that $b \leq v$; thus,

$$j_{a,v} \wedge j_{b,w} = c_b \wedge o_v = 0.$$

■

Theorem 3.5. *Suppose that L has free meets. Then $\mathbb{N}L$ is an atomic boolean frame, and L is embedded in $\mathbb{N}L$ as a complete sublattice.*

Proof. That $\mathbb{N}L$ is an atomic boolean frame follows immediately from the preceding lemma and Wilson's Theorem (Proposition 2.3(a)). L is a complete sublattice of $\mathbb{N}L$ because that embedding preserves all meets. ■

Remark 3.6. We wish to suggest that it is not enough, in general, that every non-trivial nucleus exceed an atom of the assembly, for the latter to be boolean. One must have the feature that each nucleus be a join of atoms. The following example is telling.

Let L denote the inversely well-ordered natural numbers, together with a least element 0. In [Wi94], Wilson argues that $\mathbb{N}L$ is not boolean; note that the character of 0, namely, the nucleus j defined by $j(n) = 1$, for every natural number n, and $j(0) = 0$, does not preserve the meet 0 of the natural numbers. On the other hand, for each n (which is special) $v(n) = n + 1$, and

$$j_{n,n+1}(m) = m, \quad \text{for each} \quad m \neq n + 1,$$

while $j_{n,n+1}(n + 1) = n$. And the reader will readily verify that

$$\bigvee_n j_{n,n+1}(m + 1) = m,$$

for each natural number m, so that $l = \bigvee_n j_{n,n+1} < j$. Thus, j is not a join of atoms.

The following observation further underscores the point raised just now. Proposition 3.7 is surely known; as it appears to be of such importance in the development leading up to the culminating result (Theorem 3.9), we give the short proof anyway.

Proposition 3.7. *Suppose that L is a frame in which 1 is a join of atoms. Then each nontrivial element of L is a join of atoms, and L is boolean.*

Proof. Let $A(L)$ denote the set of atoms of L; the hypothesis is that $1 = \bigvee A(L)$. So if $x \in L$ is not zero, then $x = \vee \{\, a \in A(L) \,:\, a \leq x \,\}$, by the frame law.

Next, for each $y \in L$, $y \vee y^{\perp}$ is a supremum of atoms, and hence of all the atoms; that is, $y \vee y^{\perp} = 1$. ∎

Theorem 3.3 guarantees that if L has free meets, then L is finite-valued with the descending chain condition on $\mathcal{S}(L)$, the set of special compact elements. Further, as outlined in 2.8, we have that $\mathcal{S}(L)$ is order-isomorphic to $\mathrm{Spec}(L)$ – owing to Proposition 3.1. An elementary consequence of these remarks is the following, using the notation of 2.8. We leave most of the outstanding items to be proved to the reader.

Lemma 3.8. *Assume L is a finite-valued frame, satisfying the descending chain condition on special compact elements.*

(a) *The map $a \mapsto v(a)$ induces a bijection of the atoms of L onto $\mathrm{Min}(L)$, the set of minimal primes of L.*

(b) *Each $x > 0$ in L exceeds an atom.*

(c) *Each $x \in L$, $x < 1$, is either prime or there is a value $v > x$ such that $v \to x > x$.*

Proof. (a) is obvious, and since every prime contains a minimal prime, so is (b).

As for (c), note that unless 0 is prime (in which case L is totlly ordered), there exist at least two atoms, and it is therefore clear that for some atom $a \in L$, $v(a) > 0$ as well as $v(a) \to 0 = v(a)^{\perp} > 0$. Finally, for each $x \in L$, $\uparrow x$ is algebraic with the FIP and disjointification, and also is finite-valued with the descending chain condition on $\mathcal{S}(\uparrow x)$. The claim in (c) is now also clear. ∎

We have at last arrived at the objective of the paper.

Theorem 3.9. *Suppose that L is an algebraic frame with the FIP and disjointification. Then the following are equivalent.*

(a) *L is finite-valued and satisfies the descending chain condition on compact special elements.*

(b) *L has free meets.*

(c) *L has free meets under coherent frame quotients.*

If these are satisfied, then L is completely embedded in its assembly, which is an atomic boolean frame.

Proof. The only implication we have to substantiate is that (a) implies (b). By Wilson's Theorem it suffices to show that $\mathbb{N}L$ is boolean. As to that, all that needs to be settled, according to Proposition 3.7, is that in $\mathbb{N}L$, 1 is a join of atoms. We show that

$$1 = \bigvee_{a \in \mathcal{S}(L)} j_{a,v(a)}.$$

By Lemma 1.1, it suffices to show that the pointwise supremum j_o of the $j_{a,v(a)}$ (with $a \in \mathcal{S}(L)$) has no fixed points, save 1. If $x < 1$ and prime, then by Proposition 3.1, $x = v(a)$, for a suitable special a. It is easily checked that $j_{a,x}(x) = a \vee x > x$. If x is not prime, then using Lemma 3.8(c), there is a meet-irreducible $v = v(b)$, with b special, such that both v and $v \to x$ are strictly greater than x. This implies that $b \leq v \to x$ and hence

$$j_{b,v}(x) = (b \vee x) \wedge (v \to x) = b \vee x > x,$$

since $v > x$, whence $b \not\leq x$.

In either event, the only fixed point of j_o is 1, which concludes the proof. ∎

4 Closing Remarks.

We begin this section with some questions which merit further study.

Question 4.1. *What are the consequences of the dual σ-frame law; that is, what ensues from assuming that a frame L satisfies $a \vee (\bigwedge S) = \bigwedge_{s \in S} a \vee s$, for each countable set S?*

As established by Lemma 2.4, this is enough (on an algebraic frame with disjointification) to decompose all compact elements into indecomposables.

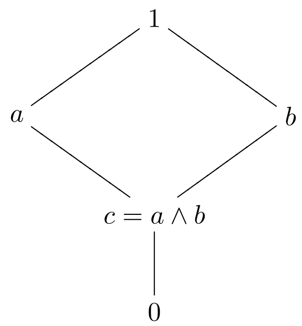
Question 4.2. *Theorem 3.5 concludes that if L is algebraic with the FIP, having disjointification and free meets, then L is a complete sublattice of an atomic boolean frame. Is the converse true?*

If L is a complete sublattice of such a boolean frame, then it too must be completely distributive ([MB89, Theorem 14.5]), and hence be a dual frame. And since atomic boolean frames are spatial, L is, perforce, spatial as well.

Next, here is a simple example showing what happens if the disjointification is discarded.

Example 4.3. Consider the distributive lattice D pictured below:

(4.3.1)



Since L is finite, it obviously has free meets. Conrad's Theorem fails: for example, 1 is indecomposable (as is every element of L), but not join-irreducible; further, 1 has no finite decomposition as join of *pairwise disjoint* special elements.

To emphasize, having free meets cannot force disjointification; nor does it follow that such a frame is normal.

The final point is formulated as a remark, rather than a question, because it is intended as a point of departure for further research.

Remark 4.4. The discussion of Section 2 takes place mostly within the framework of the Zermelo-Fraenkel axioms of set theory, whereas in Section 3 the full force of Choice is brought to bear. One is left to wonder then, whether there is a formulation of Proposition 3.1 which is Choice-free. In this regard, one would be remiss not to underscore the point already made elsewhere in this text, that the assumption in Conrad's Theorem that the frames in question here are generated by the meet-irreducible elements implies that they are spatial frames. This hypothesis implies the Boolean Prime Ideal Theorem, which is a "Choice Principle".

Looking at this kind of question from much closer, it seems appropriate to ask (about an arbitrary frame L) whether the atoms of $\mathbb{N}L$ can be described without reference to primes. Does an atom in $\mathbb{N}L$ necessarily arise as it does here; that is, if $j \in \mathbb{N}L$ is an atom, is $j = j_{a,v}$, with a join-irreducible in L and $v = a \to m(a)$, and $m(a)$ the join of all elements strictly below a? As was argued above, if j is an atom of the assembly, then $j = c_a \wedge o_q$, with $a \not\leq q$, and, furthermore, it is easy to show that if $r \in L$, such that $r \geq q$ and $a \not\leq r$, then $a \wedge r \leq q$. Thus, if q is prime, then $q = r$, and q is maximal with respect to $a \not\leq q$.

To conclude succinctly, we underscore the point made in 4.2: that a boolean frame is spatial if and only if it is atomic. Specifically, *points (primes) are the complements of atoms.* Thus, if the assembly $\mathbb{N}L$ is an atomic boolean frame, then L is spatial, and the atoms of $\mathbb{N}L$ *do* arise in the form $j = c_a \wedge o_q$, with q prime and maximal with respect to $a \not\leq q$.

But what of the algebraic features of L? The reader should note that the canonical embedding c of L in its assembly is not coherent. Further, the special building blocks of this paper are join-irreducible first, and compact second, almost *accidentally* so. One wonders then whether the algebraic assumptions carried here aren't more convenient than essential.

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Department of Mathematics, University of Florida, Box 118105
Gainesville, FL 32611-8105
email: jmartine@math.ufl.edu

Department of Mathematics and Statistics, Bowling Green State University
Bowling Green, OH 43402
email: warrenb@bgnet.bgsu.edu