Disjointifiable Lattice-Ordered Groups[1]

Jorge Martínez

Abstract. This article studies *disjointifiable* lattice-ordered groups (abbr. $d\ell$-groups): the lattice-ordered groups G for which the frame $\mathcal{C}(G)$ of all convex ℓ-subgroups is a normal frame; that is, for which $A \vee B = G$ in $\mathcal{C}(G)$ implies the existence of $C, D \in \mathcal{C}(G)$ such that $C \cap D = 0$ and $A \vee D = C \vee B = G$. It is shown that when a Hahn group $V(\Lambda, \mathbb{R})$ is a $d\ell$-group then it is *strongly* disjointifiable (abbr. $sd\ell$), in the sense that $A \vee B = G$ in $\mathcal{C}(G)$ implies that there is a cardinal summand P of G, such that $P \subseteq A$ and $P^{\perp} \subseteq B$. Every finite valued ℓ-group is an $sd\ell$-group.

As should be expected, since these concepts are intrinsically frame-theoretic, their study at the level of frames should be fruitful. Indeed, for a frame embedding $h : A \longrightarrow B$ whose adjoint satisfies the codensity condition that $a \vee b = 1$ (in B) implies that $h_*(a) \vee h_*(b) = 1$ (in A), we have that A is normal if and only if B is. Suitably interpreted for majorizing ℓ-subgroups H of G, this yields that H is a $d\ell$-group (resp. $sd\ell$-group) precisely when G has the property.

1 Introduction.

The study of disjointifiable lattice-ordered groups has very recent beginnings, and is an uncommon subject, since when the group has a strong order unit it is automatically disjointifiable, thus closing off the subject.

Without a strong unit, the question of when a lattice-ordered group (henceforth, as usual, abbreviated *ℓ-group*) G is disjointifiable, meaning that its frame of convex ℓ-subgroups $\mathcal{C}(G)$ is a normal frame, is a reasonably tricky one. This investigation is one of several independent studies of the concept. This introductory section is designed to provide the appropriate frame-theoretic backdrop for our work. Striving for efficiency, we collect this background information in the following brief commentary.

Definition & Remarks 1.1. Throughout, L is a complete lattice. The top and bottom are denoted 1 and 0, respectively. For $x \in L$, $\uparrow x$ (resp. $\downarrow x$) stands for the set of elements $\geq x$ (resp. $\leq x$). Let us also point out to the reader that, throughout, we use the phrase "y exceeds x" in a poset to indicate that $y \geq x$.

[1] **MSC Primary.** 06D22, 06F20. **Secondary.** 18B35.

Key Words. Disjointifiable ℓ-groups; normal frames; weakly zero-dimensional frames; Hahn groups.

1. $c \in L$ is *compact* if $c \leq \vee_{i\in I} x_i$ implies that $c \leq \vee_{i\in F} x_i$, for a suitable finite subset F of I. L is *algebraic* if each $x \in L$ is a supremum of compact elements. $\mathfrak{k}(L)$ stands for the set of compact elements of L. If 1 is compact it is said that L is *compact.*

2. L is said to have the *finite intersection property* (always abbreviated *FIP*) if for any pair $a, b \in \mathfrak{k}(L)$ it follows that $a \wedge b \in \mathfrak{k}(L)$. Observe that $\mathfrak{k}(L)$ is always closed under taking finite suprema. L is *coherent* if it is compact and has the FIP.

3. L is a *frame* if the following distributive law holds for each $S \subseteq L$:
$$a \wedge \left(\bigvee S\right) = \bigvee \left\{a \wedge s \,:\, s \in S\right\}.$$
It is well known that an algebraic lattice is a frame whenever it is distributive.

4. $p \in L$ is *prime* if $p < 1$ and $x \wedge y \leq p$ implies that $x \leq p$ or $y \leq p$. $\mathrm{Spec}(L)$ shall denote the set of prime elements of L.

5. Let L be a frame; $a^\perp$ denotes the supremum of all x in L for which $a \wedge x = 0$. Call $p \in L$ a *polar* if it is of the form $p = y^\perp$, for some $y \in L$. It is well known that the set PL of all polars forms a complete boolean algebra, in which infima agree with those in L.

6. The frame L is said to be *normal* if $1 = x \vee y$ implies that there exist disjoint u and v (which may be taken $u \leq x$ and $v \leq y$, respectively), such that $u \vee y = 1 = x \vee v$. If L is normal then, for each pair such that $1 = x \vee y$, u may be chosen to be a polar, with $v = u^\perp$.

   We are also interested in the following stronger condition: $1 = x \vee y$ implies that there exists a polar p in L, such that $p \leq x$ and $p^\perp \leq y$, and $1 = p \vee p^\perp$, which is to say that p is *complemented.* This condition has been studied by Banaschewski, where it is called *weak zero-dimensionality.* We shall adhere to this terminology.

7. Next, for an algebraic frame L, we highlight a property which is closely related to normality. L has *disjointification* if for each compact $c \in L$, the frame $\downarrow c$ is normal. Banaschewski, in [Ba97] and elsewhere, calls this property *coherent normality.* In [ST93] the authors call it *relative normality.* It is shown in [Mo54] that L, algebraic with the FIP, has disjointification if and only if $\mathrm{Spec}(L)$ forms a *root system*, that is, for each prime p, $(\uparrow p) \cap \mathrm{Spec}(L)$ is a chain.

8. L is *completely distributive* if for each pair of index sets I and J, and elements $x_{i,j} \in L$, we have
$$\bigwedge_{i\in I} \bigvee_{j\in J} x_{i,j} = \bigvee_{f\in J^I} \bigwedge_{i\in I} x_{i,f(i)}.$$

This concludes the catalogue of preliminaries.

Next, we record the following comment concerning frame homomorphisms.

Remark 1.2. Suppose that $h : A \longrightarrow B$ is a morphism of complete join-semilattices. Define $h_* : B \longrightarrow A$ by the following equivalence (which defines it unambiguously):

$$x \leq h_*(y) \iff h(x) \leq y.$$

The knowledgeable reader will quickly realize that, viewing A and B as categories, h_* is none other than the right adjoint of h ([HS79, 27.1]). As a consequence of the adjoint relationship, we have the following properties; the reader may also easily verify them (independent of adjoint considerations):

1. $x \leq h_*(h(x))$, for each $x \in A$, and $h(h_*(y)) \leq y$, for each $y \in B$.
2. h_* preserves arbitrary infima.

In what follows we shall use this setup when h is one-to-one; that is to say, when it is a *frame embedding*, or – dispensing with h altogether – when we wish to study the passage of the normality of a frame from a frame B to a subframe A and back.

We shall find it useful to assume that the embedding h satisfies the retractive property $h_* \cdot h = 1_A$, as well as the codensity feature that

$$x \vee y = 1 \implies h_*(x) \vee h_*(y) = 1.$$

(As the adjoint generally fails to preserve even the finite suprema, the preceding implication is, at least, a modest imposition.)

On account of the subsequent applications to ℓ-groups, we refer to such a frame embedding h as a *capping of* B *by* A, and we shall also say that A *caps* B.

Last in this frame-theoretic introduction, we record the following simple lemma, for later use.

Lemma 1.3. *Suppose that $h : A \longrightarrow B$ is a frame embedding and a capping of B by A. Then A is normal (resp. weakly zero-dimensional) if and only if B is normal (resp. weakly zero-dimensional).*

Proof. We carry out the argument for normality; it will be evident from the proof how to adapt for weak zero-dimensionality.

Suppose, first, that B is normal and that $x \vee y = 1$ in A. Then $h(x) \vee h(y) = 1$ in B, so that there exist disjoint u and v in B such that $h(x) \vee v = u \vee h(y) = 1$. Applying the adjoint h_*, we have, since A caps B, that

$$x \vee h_*(v) = h_*(1) = h_*(h(1)) = 1 = h_*(u) \vee y,$$

while $h_*(u) \wedge h_*(v) = h_*(0) = h_*(h(0)) = 0$. Thus, A too is normal.

Conversely, suppose that A is normal, and $p \vee q = 1$ in B. As A caps B, we have that $h_*(p) \vee h_*(q) = 1$. Then, owing to the normality of A, there exist disjoint $c, d \in A$ such that $c \vee h_*(q) = h_*(p) \vee d = 1$. Now apply h; one obtains

$$1 = h(c) \vee h(h_*(q)) \leq h(c) \vee q,$$

whence $h(c) \vee q = 1$, and, likewise, $p \vee h(d) = 1$, while $h(c)$ and $h(d)$ are disjoint. This shows that B is normal. ■

2 Finite Valued Frames.

The aim in this section is to prove that every completely distributive algebraic frame with the disjointification is weakly zero-dimensional. It is assumed throughout that L stands for an algebraic frame.

In [C65] Conrad characterized the ℓ-groups G for which $\mathcal{C}(G)$ is completely distributive. This is usually referred to as Conrad's Theorem on finite valued ℓ-groups. This theorem was generalized, first in [M72], and later in [ST93], to lattice-theoretic settings. Very recently, Conrad's Theorem has been reconsidered, in [M06], in frame-theoretic terms. We shall follow this development.

In fact, we state the version of Conrad's Theorem proved in [M06] within the Zermelo-Fraenkel Axioms (ZF). We supply explanations for the undefined terms in the following commentary, as indicated in the theorem.

Theorem 2.1. *For an algebraic frame L with the disjointification, the following are equivalent:*

1. $\mathrm{Val}(L)$ *generates L, and each value of L is special.* (See 2.2.2.)
2. $\mathrm{Val}(L)$ *freely generates L.* (See 2.2.4.)
3. *L is completely distributive.*
4. *The dual frame law holds L; that is, for each $a \in L$ and subset S of L,*

$$a \vee \left(\bigwedge S \right) = \bigwedge \left\{ a \vee s \; : \; s \in S \right\}.$$

5. *Each $a \in \mathfrak{k}(L)$ can be decomposed, uniquely, as a finite supremum of pairwise disjoint finitely join-irreducible elements.* (See 2.2.3.)

Definition & Remarks 2.2. As in Theorem 2.1, L stands for an algebraic frame L with the disjointification.

1. If $v \in L$ is maximal with respect to not exceeding some $c \in \mathfrak{k}(L)$, then we say that v is a *value* of c. It is well known that values are prime.

2. If c has exactly one value v, then we say c and v are *special*, and also that c is *special at v*.

3. One of the consequences of Theorem 2.1 – already remarked in [M06] – is that a compact element c is special if and only if it is *finitely join irreducible*: $c \leq x \vee y$ implies that $c \leq x$ or $c \leq y$.

4. We denote the set of all values of L by $\mathrm{Val}(L)$. We say that $\mathrm{Val}(L)$ *generates* L if each $x \in L$ is a meet of values. If for any two up-sets S_1 and S_2 of values,

$$\bigwedge S_1 = \bigwedge S_2 \implies S_1 = S_2,$$

we say that $\mathrm{Val}(L)$ *freely generates* L. (If P is any poset and $S \subseteq P$, we say that S is an *up-set* if $y \geq x \in S$ implies that $y \in S$. A *down-set* is defined dually.)

Finally, we call L *finite valued* if it satisfies the conditions of Theorem 2.1.

The following corollary greatly facilitates matters. We underscore what should be obvious to the knowledgeable frame-theorist: that, while Theorem 2.1 builds in the spatial property of the frames in question, the Axiom of Choice is not used in its proof. The claims of the corollary, likewise, while requiring the application of Zorn's Lemma in most developments, here are obtained from the preceding theorem; that is, Choice-free.

For each special compact element a, let v_a denote its value, and $\mathfrak{s}(L)$ stand for the set of (nonzero) special compact elements of L

All the assertions in this corollary are routine, and they are left to the reader.

Corollary 2.3. *Suppose L is a finite valued frame with disjointification. Then we have the following.*

(a) *The map $a \mapsto v_a$ is an order isomorphism from $\mathfrak{s}(L)$ onto $\mathrm{Val}(L)$.*

(b) *If $a, b \in \mathfrak{s}(L)$, then either they are comparable or disjoint.*

(c) *$\mathfrak{s}(L)$ is a root system.*

We are now ready for the theorem promised at the beginning of this section.

Theorem 2.4. *Suppose that L has disjointification and is finite valued. Then it is also weakly zero-dimensional. More precisely, if $1 = x \vee y$, then there exist complemented elements p and q such that*

(a) *$p, q^{\perp} \leq y$ and $q, p^{\perp} \leq x$, and*

(b) *p and q are least for (a); that is, if p_1 and q_1 are complemented elements such that $p_1, q_1^{\perp} \leq y$ and $q_1, p_1^{\perp} \leq x$, then $p \leq p_1$ and $q \leq q_1$.*

Proof. First, as L is finite valued, each element $a \neq 0$ in L is a supremum of special elements. It also clearly suffices to consider decompositions $1 = x \vee y$ in which $x, y < 1$.

Consider a pair of special compact elements a and b, such that $a \not\leq x$ and $b \not\leq y$. Then $x \leq v_a$ and $y \leq v_b$, so that $v_a \vee v_b = 1$ and the values v_a and v_b are incomparable. By Corollary 2.3, this implies that $a \wedge b = 0$. Note, in particular, that this implies that for any $s \in \mathfrak{s}(L)$, either $s \leq x$ or $s \leq y$.

Now, let $p = \vee \{ a \in \mathfrak{s}(L) : a \not\leq x \}$ and $q = \vee \{ a \in \mathfrak{s}(L) : a \not\leq y \}$. We verify that p is complemented and that it and $p^\perp$ witness the weak zero-dimensionality of L for x and y. An identical argument will apply to q. Note for the record that $p \wedge q = 0$, by the results of the preceding paragraph.

For suppose that $p \vee p^\perp < 1$, and let s be a special compact element such that $s \not\leq p \vee p^\perp$. Then $s \wedge p > 0$, whence we have $a \in \mathfrak{s}(L)$ with $a \not\leq x$ such that $s \wedge a > 0$. According to Corollary 2.3 this means that s and a are comparable, forcing $s \leq p$, which is a contradiction. Thus p is complemented.

Next, it is clear that $p \leq y$, and if $p^\perp \not\leq x$, then we have a nonzero special compact element b such that $b \wedge p = 0$, yet $b \not\leq x$; but the latter condition forces $b \leq p$, which is absurd. Therefore, $p^\perp \leq x$, and this completes the proof that L is weakly zero-dimensional.

Condition (b) of the theorem remains; suppose that p_1 and q_1 are complemented elements satisfying the premise of (b). Observe that if $a \in \mathfrak{s}(L)$ and $a \not\leq x$ then, since $p_1 \vee x = 1$, we have that $a \leq p_1$, whence $p \leq p_1$, and similarly, $q \leq q_1$. ■

Here is an immediate consequence of Lemma 1.3 and the preceding theorem.

Corollary 2.5. *Let $h : A \longrightarrow B$ be a capping frame embedding, with A algebraic and finite valued with disjointification. Then B is weakly zero-dimensional.*

We conclude this section with a result which settles a question posed indirectly in [M06]: why the FIP does not play a role in Theorem 2.1.

Proposition 2.6. *Suppose L is a finite valued frame with disjointification. Then the FIP is also satisfied.*

Proof. In view of Theorem 2.1 and the distributive law, it suffices to show that the meet of two special compact elements is compact. This is immediate from Corollary 2.3(b). ■

3 Capping ℓ-Subgroups.

In this section we set up the appropriate ℓ-group-theoretical setting, where we shall consider applications of Lemma 1.3. Let us begin by recalling some basic features of the frame $\mathcal{C}(G)$ of all convex ℓ-subgroups of the ℓ-group G. Our standard references for ℓ-groups are [BKW77] and [D95].

Definition & Remarks 3.1. Throughout this commentary, G denotes an arbitrary ℓ-group. We write every ℓ-group additively. G^+ stands for the cone of positive elements of G, and for each $g \in G$, one writes $g^+ = g \vee 0$, $g^- = (-g) \vee 0$, and $|g| = g \vee (-g)$; these are the *positive part*, the *negative part*, and the *absolute value* of g, respectively. Also observe that $|g| = g^+ \vee g^- = g^+ + g^-$; (see [D95, §4].)

Recall that $A \subseteq G$ is an *ℓ-subgroup* of G if it is at once a subgroup and a sublattice of G. Note that A is an ℓ-subgroup of G if and only if $g \in A$ implies that $g^+ \in A$. A subgroup A of G is *convex* if $0 \le x \le a \in A$ implies that $x \in A$. Note that the subgroup A of G is a convex ℓ-subgroup if and only if $|x| \le |a|$ and $a \in A$ together imply that $x \in A$ ([D95, Proposition 7.1]).

$\mathcal{C}(G)$ denotes the set of all convex ℓ-subgroups of G. $\mathcal{C}(G)$ is a complete sublattice of the lattice of subgroups of G ([D95, Theorem 7.5]), and a frame, due to Birkhoff's Theorem ([D95, Proposition 7.10]). We shall depart from the standard custom by denoting the convex ℓ-subgroup generated by $S \subseteq G$ by $\langle S\rangle_c$, and when necessary to avoid confusion, use $\langle S\rangle_c^G$ to stipulate that it is in G that the subgroup is generated. Observe that if S is already a subgroup, then

$$\langle S\rangle_c = \{\, g \in G \,:\, |g| \le |s|, \text{ for suitable } s \in S \,\}.$$

In the case of $S = \{x\}$, a singleton set, we write $\langle x\rangle_c$ for $\langle \{x\}\rangle_c$.

Call a subgroup S of G *majorizing* if $G = \langle S\rangle_c$.

Observe that $\mathcal{C}(G)$ is an algebraic frame in which A is compact if and only if it is principally generated, i.e., when $A = \langle x\rangle_c$, for some $x \in A$ ([D95, Proposition 7.16]). $\mathcal{C}(G)$ is a frame with the FIP ([D95, Proposition 7.15]) and disjointification ([D95, Corollary 9.3]).

Finally, some notational observations: the boolean algebra of polars of $\mathcal{C}(G)$ is denoted $\mathcal{P}(G)$, and we shall keep to the convention in the literature on ℓ-groups by writing $x^\perp$ for $\langle x\rangle_c^\perp$, for each $x \in G$.

Next, we discuss how a majorizing ℓ-subgroup H allows us to view $\mathcal{C}(H)$ as a subframe of $\mathcal{C}(G)$.

Definition & Remarks 3.2. Let H stand throughout this commentary for a fixed ℓ-subgroup of the ℓ-group G. We denote the map $C \mapsto \langle C\rangle_c^G$, from $\mathcal{C}(H)$ to $\mathcal{C}(G)$ by γ. Let us observe the following properties of γ; the proofs are all straightforward.

1. γ preserves arbitrary suprema; this is a consequence of the Riesz Interpolation Property of ℓ-groups ([D95, Theorem 3.11]): *in any ℓ-group, $a \le b_1 + \cdots + b_n$, for positive elements, implies that there exist positive $a_1, \ldots, a_n$, such that $a_i \le b_i$ ($1 \le i \le n$) and $a = a_1 + \cdots + a_n$.*

2. γ preserves finite intersection.

3. γ_* is the trace map $A \mapsto A \cap H$, and $\gamma_* \cdot \gamma = 1_{\mathcal{C}(H)}$.

4. $\gamma(0) = 0$, and $\gamma(H) = G$ precisely when H majorizes G

Putting together the above four properties we have that γ is a frame embedding, as long as H majorizes G. γ is a capping of $\mathcal{C}(G)$ by $\mathcal{C}(H)$ if and only if

$$G = A \vee B \text{ (in } \mathcal{C}(G)) \implies H = (A \cap H) \vee (B \cap H) \text{ (in } \mathcal{C}(H)). \tag{3.2.1}$$

When this is the case, we shall say that H is a *capping of* G, or that H *caps* G.

Before applying the results of the preliminaries on frames, let us make the connection between finite valued frames and ℓ-groups.

Remark 3.3. Let G denote an ℓ-group. We describe the setting of Conrad's original work in [C65].

We say that G is *finite valued* if $\mathcal{C}(G)$ is finite valued. If $0 \neq g \in G$, then we also call g *special* when $\langle g \rangle_c$ is special in $\mathcal{C}(G)$. The reader will note that $V \in \mathcal{C}(G)$ is a value of $\langle g \rangle_c$ precisely when V is maximal with respect to excluding g. As noted in 2.2, all values in $\mathcal{C}(G)$ are prime. We shall abbreviate $\mathrm{Val}(\mathcal{C}(G)) \equiv \mathrm{Val}(G)$.

Note that each $V \in \mathrm{Val}(G)$ is covered in $\mathcal{C}(G)$ by

$$V^* \equiv \bigcap \Big\{ C \in \mathcal{C}(G) \,:\, V \subset C \Big\},$$

which properly contains V and is obviously the least such.

For use in §4, let us also recall the convention that when $V \in \mathcal{C}(G)$ is maximal with respect to excluding $g \in G$, that V is said to be a value of g.

We now formally record the consequences of Lemma 1.3 and the results of §2.

Proposition 3.4. *Let H be a capping ℓ-subgroup of G. Then $\mathcal{C}(G)$ is a normal (resp. weakly zero-dimensional) frame if and only if $\mathcal{C}(H)$ is normal (resp. weakly zero-dimensional). In particular, if G has a finite valued cap then $\mathcal{C}(G)$ is weakly zero-dimensional.*

For the context of ℓ-groups, we establish the following terminology.

Definition 3.5. An ℓ-group G is said to be *disjointifiable* (resp. *strongly disjointifiable*) if $\mathcal{C}(G)$ is normal (resp. strongly disjointifiable). The reader should not fear, as we will, for the most part, use the abbreviated designations *$d\ell$-group* and *$sd\ell$-group* for "disjointifiable" and "strongly disjointifiable" ℓ-groups, respectively.

To end this section, we refer the reader to Example 5.8, which is not a $d\ell$-group, and which also points out how crucial the role is of the majorizing feature in a capping ℓ-subgroup. This example appears in [MZ06], with a different purpose in mind.

4 Semidirect Factors which Lie Above.

In this section we apply the foregoing, by identifying semidirect factors of an ℓ-group which also cap the parent group. In the following we shall couch the proceedings in terms of semidirect products, and so we begin with a preamble on that subject.

Definition & Remarks 4.1. Throughout this commentary G denotes an ℓ-group, and we have ℓ-subgroups A and H of G, such that A is also normal and convex, and

1. $A \cap H = \{0\}$ and $G = A + H$, while
2. the projection $\pi(a + x) = x$ (with $a \in A$ and $x \in H$) is an ℓ-homomorphism.

When we use the phrase below, "G is a semidirect product of subgroups A and H", we shall always assume the aforementioned properties. The order of mention too is fixed: A will always be the normal factor in $\mathcal{C}(G)$, and H the not-necessarily-normal (and typically far from convex) factor.

With regard to the arithmetic in a semidirect product of A and H, note that each $g \in G$ can be expressed uniquely as $g = a + h$, with $a \in A$ and $h \in H$, and, for each $a \in A$ and $h \in H$, there is a unique $b \in A$, such that $(a + h)^+ = b + h^+$.

We will identify a class of semidirect products G of A and H, such that H caps G. For this we need to review the principle of an "element lying above" another; this is a technical concept, first introduced in [BCD86] by Ball, Conrad, and Darnel. These ideas are closely related to the split subgroups of [M90]. The reader is referred to these two papers, and also to [D95, §20], for additional background on this subject.

The reader might well begin to wonder, reading on, whether much of the discussion might not be carried out at the level of algebraic frames. This is indeed the case, except that frame-theoretically, the normality of subgroups is difficult to track. As the reader who is knowledgeable about ℓ-groups is aware, a value V is not necessarily normal in the group, and not even normal in its cover V^*. Since this issue rears its head in the upcoming discussion on "lying above", we are left with the choice of taking the matter into account, or else limiting the conversation to abelian ℓ-groups. We have chosen the first course, even though most of the applications we have in mind occur in abelian groups.

For the record, if each value V of G is normal in its cover V^*, it is said that G is *normal valued*.

Definition & Remarks 4.2. Let G denote an ℓ-group throughout this commentary. Recall at the outset that it is customary, for positive $a, b \in G$, to write $a \ll b$ when $na < b$, for each $n \in \mathbb{N}$; we then say that a is *infinitesimal* to b.

We say that $a \in G$ *lies above* $b \in G$ – and then also that b *lies below* a – if $|a| \wedge |b| \ll |a|$. From [BCD86, Proposition 1.1], a lies above b precisely when there is no prime P in $\mathcal{C}(G)$ and no integer n such that

$$P < P + |a| < P + n|b|.$$

Using this, it is easy to show that if every value of a contains b, then a lies above b, and the converse holds if G is normal valued.

If A and B are subgroups of G, one says that A *lies above* B (or that B *lies below* A) if each element of A lies above B. One then has the following.

Proposition 4.3. ([BCD86, Proposition 1.4]) *The following are equivalent for subgroups A and B of G.*

(a) *A lies above B.*

(b) *$a^+ \wedge (b-a)^+ = 0$, for each $a \in A$ and $b \in B$.*

If G is normal valued then the above are equivalent to

$$B \subseteq \mathrm{bl}(A) \equiv \bigcap \Big\{ V \in \mathrm{Val}(G) \,:\, V \text{ is a value of some } a \in A \Big\}.$$

Remark 4.4. For obvious reasons we shall refer to $\mathrm{bl}(A)$, as defined above, as the *subgroup below* A. Notice that, for any subgroup A of G, $\mathrm{bl}(A) \in \mathcal{C}(G)$. It should be clear that

1. $A \cap \mathrm{bl}(A) = \{0\}$, and
2. if A is normal in G, then so is $\mathrm{bl}(A)$.

Indeed, one may similarly define this kind of construct in any algebraic frame L; for any subset $A \subseteq \mathfrak{k}(L)$, put

$$\mathrm{bl}(A) = \wedge \{ v \in \mathrm{Val}(L) \,:\, v \text{ is a value of some } a \in A \}.$$

A study of this construct in frames will be taken up elsewhere.

Returning to semidirect products, we record the following result. It is essentially [D95, Proposition 20.12]; however, there is more here than might first meet the eye: we therefore sketch the proof of condition 4.1.2 below.

Lemma 4.5. *Suppose that H is an ℓ-subgroup of G, such that G is the subgroup generated by $\mathrm{bl}(H)$ and H, and $\mathrm{bl}(H)$ is normal in G. Then G is a semidirect product of $\mathrm{bl}(H)$ and H.*

Proof. **(Sketch.)** The identity crucial to understanding why the projection $a + h \mapsto h$ is an ℓ-homomorphism is the following; see [D95, Proposition 20.10] or [BCD86, Lemma 3.1]. For each $h \in H$ and $x \in \mathrm{bl}(H)$,

$$(x+h)^+ = h^+ + (x^+ \wedge h^+) - (x^- \wedge h^+) + (\pi_h(x))^+, \tag{4.5.1}$$

where $\pi_h : \langle x\rangle_c \longrightarrow h^\perp$ denotes the projection on $\langle x\rangle_c \cap h^\perp$ which is given by the decomposition

$$\langle x\rangle_c = (\langle x\rangle_c \cap \langle h\rangle_c) \oplus (\langle x\rangle_c \cap h^\perp),$$

which is a consequence of [BCD86, Proposition 1.1]. Finally, it should be clear that equation (4.5.1) shows that the projection $a+h \mapsto h$ is an ℓ-homomorphism: the point to make in this regard is that in (4.5.1) the terms in $(x^+ \wedge h^+) - (x^- \wedge h^+) + (\pi_h(x))^+$ all lie in $\mathrm{bl}(H)$. ■

We are now able to reconnect with caps and $d\ell$-groups.

Theorem 4.6. *Suppose that G is an ℓ-group and H is an ℓ-subgroup such that* $\mathrm{bl}(H)$ *is a normal subgroup of G and $G = \mathrm{bl}(H) + H$. As long as H majorizes G, then H caps G, and so H is a $d\ell$-group (resp. an $sd\ell$-group) if and only if G is a $d\ell$-group (resp. an $sd\ell$-group).*

Proof. In view of Proposition 3.4 and the assumption that H majorizes G, all that needs to be established is condition 3.2.1. To that end, suppose that $G = A \vee B$, with $A, B \in \mathcal{C}(G)$, and consider $h = \sum_{i=1}^n a_i + b_i \in H$, with each $a_i \in A$ and each $b_i \in B$.

Write each $a_i = x_i + y_i$ and $b_i = u_i + v_i$, with $x_i, u_i \in \mathrm{bl}(H)$ and $y_i, v_i \in H$, for each $i = 1, \ldots, n$. Then

$$h = c + \Big(\sum_{i=1}^n (y_i + v_i)\Big),$$

for suitable $c \in \mathrm{bl}(H)$. From this we conclude that $c = 0$. Thus, $h = \sum_{i=1}^n (y_i + v_i)$, which is an expression in the group H. It is enough to verify that each $y_i \in A$, while each $v_i \in B$. The argument for the y_i will plainly apply, *mutatis mutandis*, to the v_i.

Suppose that some $y_i \notin A$; then there is a value V of y_i, containing A. Now, the reader will observe that $a_i = x_i + y_i \in A \subseteq V$, and since $x_i \in V$, because it lies in $\mathrm{bl}(H)$, and $y_i \in H$, we have that $y_i = -x_i + a_i \in V$, which is absurd. Thus, $y_i \in A$, and the proof is complete. ■

5 Hahn Groups.

The question posed in this section is: *When is a Hahn group a $d\ell$-group?* We don't know the complete answer, but we are able to apply Theorem 4.6 to arrive at a reasonable necessary condition on the underlying root system for such a group to be a $d\ell$-group (Theorem 5.5), and this is used to show that whenever a Hahn group is a $d\ell$-group, then it is, in fact, an $sd\ell$-group (Theorem 5.7). Let us first recall what a Hahn group is. In the sequel, whenever we speak of a *real group*, we mean a subgroup of the additive real numbers with the usual ordering. We shall (for convenience) always assume that real groups contain the number 1; no generality is lost in the sequel on this account.

Definition & Remarks 5.1. Let Λ denote a fixed partially ordered set, and $\{ R_\lambda : \lambda \in \Lambda \}$ be a set of real groups. Now, $V = V(\Lambda, \mathbb{R}_\lambda)$ stands for the subgroup of the direct product $\prod_{\lambda\in\Lambda} R_\lambda$ consisting of all functions f for which

$$\mathrm{coz}(f) \equiv \{ \lambda \in \Lambda \,:\, f(\lambda) \neq 0 \}$$

satisfies the ascending chain condition. A partial ordering is defined on V by: $f > 0$ if $f(\mu) > 0$ for each maximal element $\mu \in \mathrm{coz}(f)$. It is well known that V is thereby lattice-ordered if and only Λ is a root system ([D95, Theorem 51.3]). The fundamental Conrad-Harvey-Holland Theorem – see [D95, Corollary 51.9] – shows that every abelian ℓ-group may be embedded in $V(\Lambda, \mathbb{R}_\lambda)$, for a suitable root system Λ and real groups R_λ.

It will be useful to establish some additional terminology regarding the underlying root system of a Hahn group.

Definition & Remarks 5.2. Suppose that Λ is a root system. The elements $\lambda, \mu \in \Lambda$ are *linked* if they have a common upper bound. Linkage defines an equivalence relation, the equivalence classes of which are called *components* of Λ. We shall say that Λ is *connected* if it has only one component.

We say that a component Δ of Λ is *lexed* if there is a $\delta \in \Delta$ which is comparable to every element in Δ. If δ witnesses this property for Δ, we call δ a *lexing for* Δ. We let $l(\Delta)$ denote the set of all lexings for Δ; note that $l(\Delta) = \emptyset$ if Δ is not lexed. Observe that the union, $\widehat{\Lambda}$, of all the $l(\Delta)$ is a root system in which all the components are chains; if each component of Λ is lexed, then we shall refer to $\widehat{\Lambda}$ as the *lexed capping* of Λ.

Suppose that every component of Λ is lexed, and let $\widehat{\Lambda}$ be its lexed capping. Suppose that $\{ R_\lambda : \lambda \in \Lambda \}$ is a set of real groups. Then $V(\widehat{\Lambda}, R_\lambda)$ is canonically ℓ-isomorphic to the ℓ-subgroup K_Λ of $V = V(\Lambda, R_\lambda)$ defined by

$$K_\Lambda \equiv \{ f \in V \,:\, f(\lambda) = 0,\ \lambda \notin \widehat{\Lambda} \}.$$

Thus, K_Λ can be seen as the ℓ-subgroup of all functions which vanish at all the non-lexings of Λ. The task in the proof of Theorem 5.7 below is to show that K_Λ is a cap of V.

The root systems depicted in the proof of Theorem 5.5 and in Example 5.8 are connected, and neither is lexed.

In advance of Theorem 5.5, we revert, briefly, to the setting of frames for a lemma. The proof is straightforward and is left to the reader.

Lemma 5.3. *Let L be a frame and $a \in L$ be complemented. Then L is normal (resp. weakly zero-dimensional) if and only if $\uparrow a$ and $\uparrow a^\perp$ both are normal (resp. weakly zero-dimensional).*

Lemma 5.3 has the following consequence for Hahn groups.

Corollary 5.4. *Let Λ be a root system, and suppose that $\Lambda = \Lambda_1 \cup \Lambda_2$ is a partition of Λ into two unions of its components. Then $V(\Lambda, R_\lambda)$ is a $d\ell$-group (resp. an $sd\ell$-group) if and only if both $V(\Lambda_1, R_\lambda)$ are $V(\Lambda_2, R_\lambda)$ $d\ell$-groups (resp. $sd\ell$-groups).*

Proof. Identify $V_1 = V(\Lambda_1, R_\lambda)$ with the convex ℓ-subgroup of $V = V(\Lambda, R_\lambda)$ of all functions that vanish of Λ_2 and $V_2 = V(\Lambda_2, R_\lambda)$ with the convex ℓ-subgroup of all functions that vanish on Λ_1. Then observe that $V_1^\perp = V_2$ and that $V = V_1 + V_2$, and apply the lemma. ■

Theorem 5.5. *Suppose that $V(\Lambda, R_\lambda)$ is a $d\ell$-group. Then every component of Λ is lexed.*

Proof. Thanks to Corollary 5.4 we may assume that Λ is connected. Now assume that Λ is not lexed. Then it has a cofinal subset Δ consisting of two transfinite sequences $\{\gamma_\alpha : \alpha < \beta\}$ and $\{\delta_\alpha : \alpha < \beta\}$, with

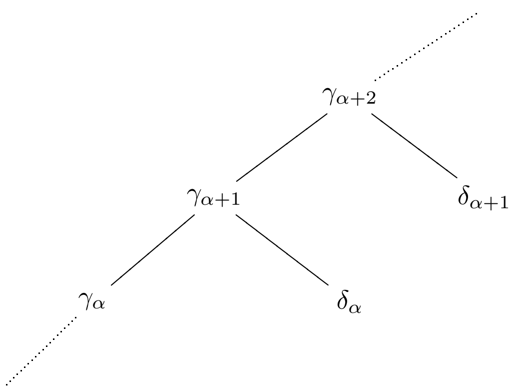
(i) $\{\gamma_\alpha : \alpha < \beta\}$ is a chain such that $\alpha < \alpha' < \beta$ implies that $\gamma_\alpha < \gamma_{\alpha'}$;

(ii) $\{\delta_\alpha : \alpha < \beta\}$ is an antichain;

(iii) $\gamma_\alpha, \delta_\alpha < \gamma_{\alpha+1}$, for each $\alpha < \beta$.

Here is a sketch of Δ.

(5.5.1)



We exhibit a pair of convex ℓ-subgroups of V that witnesses the fact that V is not disjointifiable. Let e_α denote the characteristc function of the singleton $\{\gamma_\alpha\}$, and

$$A = \langle \{ e_\alpha : \alpha < \beta \} \rangle_c, \quad \text{and} \quad B = \{ f \in V : f(\lambda) = 0, \text{ for each } \lambda \geq \gamma_\alpha, \ \alpha < \beta \}.$$

Before proceeding any further, note that if $\lambda \in \Lambda$ exceeds some γ_α then, for each $\alpha < \alpha' < \beta$, we have either $\lambda \le \gamma_{\alpha'}$ or else the reverse inequality. By the definition of the ordering on V, it also follows that B is a convex ℓ-subgroup of V. We note, moreover, that both A and B are proper subgroups.

We claim, first, that $V = A + B$. For suppose that $0 \le f \in V$. If $f \in B$ there is nothing to prove, so we assume that $f \notin B$. Then there is a $\gamma \in \mathrm{coz}(f)$, maximal in $\mathrm{coz}(f)$, exceeding some γ_α. Moreover, this γ is uniquely determined, by the comment following the definitions of A and B. Now let

$$g(\mu) = \begin{cases} f(\mu) & \text{if } \mu \le \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g \in V$, and since Δ is cofinal, it is also easily seen that $g \in A$. On the other hand, $f - g \in B$, proving the claim.

Next, suppose, by way of contradiction, suppose that there exist $C, D \in \mathcal{C}(V)$, such that $C \cap D = \{0\}$, and $C \subseteq A$, $D \subseteq B$, with $A + D = V = C + B$. Then, in the same spirit as the argument in the second paragraph of the proof of Theorem 2.4, for each $\alpha < \beta$, we have that either $e_\alpha \in C$ or $e_\alpha \in B$ – because e_α is special. Since the second option is clearly impossible, we conclude that each $e_\alpha \in C$, and thus, $A = C$, which means that $V = A \oplus D$ and $A^\perp = D$. However, due to the cofinality of Δ in Λ, $A^\perp = \{0\}$, whence $V = A$, which is absurd. Therefore no such convex ℓ-subgroups exist, and we have shown that V is not disjointifiable, as promised. ∎

To use Theorems 4.6 and 5.5 in the proof of the next theorem, we need the following lemma. Recall that an ℓ-group G is *strongly projectable* if $G = P \vee P^\perp$, for each polar P of G; that is, if, in the frame $\mathcal{C}(G)$, every polar is complemented.

Lemma 5.6. *If G is strongly projectable and a $d\ell$-group, then it is also an $sd\ell$-group.*

Proof. If $G = A \vee B$, with $A, B \in \mathcal{C}(G)$, then there is a polar P such that $P \subseteq A$ and $P^\perp \subseteq B$, and $G = A \vee P^\perp = P \vee B$. As G is also $P \vee P^\perp$, we conclude that G is an $sd\ell$-group, as claimed. ∎

Theorem 5.7. *For each root system Λ, whenever $V(\Lambda, R_\lambda)$ is a $d\ell$-group, then it is also an $sd\ell$-group.*

Proof. Let $V = V(\Lambda, R_\lambda)$, and suppose it is disjointifiable. Then, by Theorem 5.5, Λ has lexed components. So $V(\widehat{\Lambda}, R_\lambda) \cong K_\Lambda$ (5.2) is a product of totally ordered groups – since the underlying root system $\widehat{\Lambda}$ is a disjoint union of chains – and, therefore, K_Λ is strongly projectable.

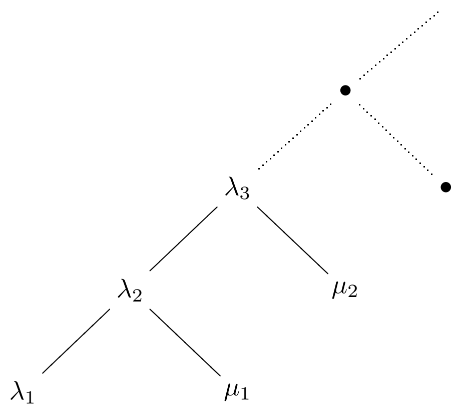
On the other hand, the reader will easily verify that

1. $\mathrm{bl}(K_\Lambda) = \{\, f \in V \;:\; f|_{\widehat{\Lambda}} \equiv 0 \,\}$;
2. K_Λ majorizes V;

3. V is the semidirect product of $\mathrm{bl}(K_\Lambda)$ and K_Λ.

Applying Theorem 4.6, we conclude that K_Λ is a $d\ell$-group, and therefore, by Lemma 5.6, an $sd\ell$-group. Using Theorem 4.6 in the reverse direction, we finally conclude that V is strongly disjointifiable. ■

Example 5.8. Consider the Hahn group $V = V(\Lambda, \mathbb{R})$ over the root system Λ described below.



$G \subseteq V$ is defined to be the group consisting of all functions f such that $f(\lambda_n)$ is eventually zero and $f(\mu_n)$ is eventually constant. (Note that a sequence $(r_n)_{n<\omega}$ of real numbers is *eventually* r provided there is a natural number k, such that, for all $n \geq k$, we have that $r_n = r$; such sequences are said to be *eventually constant.*)

There are two points to be made:

1. *G is not a $d\ell$-group.* This is witnessed by the following pair of convex ℓ-subgroups: let A consist of all $f \in G$ such that $f(\mu_n)$ too is eventually zero, and B consist of all $f \in G$ such that $f(\lambda_n)$ is identically zero. Clearly, $G = A + B$. The details involved in checking that A and B do indeed witness what is claimed are similar to those of the proof of Theorem 5.5, and are left to the reader.

2. *Let H be a the ℓ-subgroup of all $f \in G$ for which $f(\mu_n)$ is identically zero. Then H does not majorize G.*

   However, if $G = C + D$, with $C, D \in \mathcal{C}(G)$, then as argued for finite valued frames, each $h \in H$ lies either in C or D, since H consists of special elements of G. Indeed, let h_n be the function in H such that $h_n(\lambda_m) = \delta_{nm}$. If $H \not\subseteq D$, there is a least m such that $h_m \notin D$, and then it follows that $h_n \notin D$, for each $n \geq m$. This implies that $h_n \in C$ for each $n \in \mathbb{N}$, whence $H \subseteq C$.

Thus, H is trivially $(H \cap C) + (H \cap D)$, proving that 3.2.1 holds, without the majorizing feature. Yet H is totally ordered, and therefore an sdℓ-group, whereas G is not even dℓ.

6 Special Valued ℓ-Groups.

What really makes Theorem 5.5 work? We answer that, presently, by generalizing said theorem. The proper context is that of special valued ℓ-groups, which we now recall.

Definition & Remarks 6.1. Throughout these remarks it is assumed that G denotes an ℓ-group. Recall that $\mathrm{Val}(G)$ stands for the set of values of G; denote by $\mathrm{SpVal}(G)$ the subset of all special values – that is, the set of special elements of $\mathrm{Val}(G)$. Part of the thrust of Theorem 2.1 is that G is finite valued precisely when $\mathrm{Val}(G) = \mathrm{SpVal}(G)$.

Now we say that G is *special valued* if each $0 \leq g \in G$ can be expressed as $g = \vee_{i \in I} s_i$, with each $s_i \geq 0$ special, and $s_i \wedge s_j = 0$, for each $i \neq j$. It is well known that G is special valued if and only if $\mathrm{SpVal}(G)$ is a *plenary* subset of values; that is, $\bigcap \mathrm{SpVal}(G) = \{0\}$ and $\mathrm{SpVal}(G)$ is up-closed in $\mathrm{Val}(G)$: $W \subseteq V$ and $W \in \mathrm{SpVal}(G)$, with $V \in \mathrm{Val}(G)$, imply that $V \in \mathrm{SpVal}(G)$ ([D95, Theorem 45.6]).

When G is special valued, and $g \in G$ is expressed as $g = \vee_{i \in I} s_i$, with each $s_i \geq 0$ special, and $s_i \wedge s_j = 0$, for each $i \neq j$, then this expression is unique. Each s_i is referred to as a *special component*.

It is shown in [D95, Proposition 51.4] that every Hahn group is special valued.

Whereas the following lemma could have been established much earlier, we feel that it more properly belongs here, in part because stating it involves the lex-kernel of an ℓ-group, a notion which we now recall. The reader should refer to [D95, §27].

Remarks 6.2. Let G denote an ℓ-group. There is a convex ℓ-subgroup, denoted $\mathrm{lex}(G)$, which is the least $C \in \mathcal{C}(G)$ which is comparable to every convex ℓ-subgroup. $\mathrm{lex}(G)$ is called the *lex-kernel* of G. The lex-kernel can be characterized in several ways:

1. $\mathrm{lex}(G)$ is the supremum in $\mathcal{C}(G)$ of all the minimal primes of $\mathcal{C}(G)$. ([D95, Proposition 27.2])
2. $\mathrm{lex}(G)$ is the supremum in $\mathcal{C}(G)$ of all the proper polars. ([D95, Proposition 27.9])
3. $\mathrm{lex}(G)$ is the subgroup of G generated by all nonunits of G. ([D95, Proposition 27.12]) ($0 < x \in G$ is a *nonunit* if there exists a $0 < y \in G$ such that $x \wedge y = 0$.)

Observe as well that if $s \in G \setminus \mathrm{lex}(G)$, then s is special.

Lemma 6.3. *Suppose that* $\mathrm{lex}(G)$ *is a proper subgroup of* G. *Then* G *is an sdℓ-group.*

Proof. Suppose that $G = A \vee B$ in $\mathcal{C}(G)$, and note, as in the proof of Theorem 2.4, that if $s \in G$ is special then $s \in A$ or $s \in B$. So suppose that some special $0 < s \notin \mathrm{lex}(G)$ fails to be in A; then if $t \geq s$, it follows that t too is special and $t \notin A$, whence $\uparrow s \subseteq B$. Since the convex hull of $\uparrow s$ is G, we have that $G = B$. Thus the trivial polar pair G and $\{0\}$ witness the strong disjointification of B and A. ∎

We are able to generalize Theorem 5.5 as follows, in two stages.

Proposition 6.4. *Suppose that G is special valued, and assume that* $\mathrm{SpVal}(G)$ *is connected. Then G is disjointifiable if and only if* $\mathrm{SpVal}(G)$ *is lexed. If this is the case then G is also strongly disjointifiable.*

Proof. For the sufficiency, note that if $\Lambda = \mathrm{SpVal}(G)$ is lexed, and $\widehat{\Lambda}$ is the lexed capping of Λ, then $\mathrm{lex}(G)$ is the intersection of all the values $V \in \widehat{\Lambda}$, and is a proper subgroup of G. By Lemma 6.3 G is an $sd\ell$-group.

Now the necessity of the condition: suppose, by way of contradiction, that Λ is not lexed. We abuse the notation by indexing the special values by Λ; i.e., $\mathrm{SpVal}(G) = \{ V_\lambda : \lambda \in \Lambda \}$. Then, as in the proof of Theorem 5.5 it has a cofinal subset Δ as depicted in (5.5.1). We continue to mimic the proof of Theorem 5.5: let

$$A = \langle \{ e_\alpha : \alpha < \beta \} \rangle_c, \quad \text{and} \quad B = \bigcap \Big\{ V_\lambda : \ \text{for each } \lambda \geq \gamma_\alpha, \ \alpha < \beta \Big\},$$

where, for each $\alpha < \beta$, $e_\alpha > 0$ is a special element having its value at V_{γ_α}. Once more, these convex ℓ-subgroups witness that G is not a $d\ell$-group. For, on the one hand, $G = A + B$: if $0 < g \notin B$, then g has a special value V_λ, for some $\lambda \geq \gamma_\alpha$ (for suitable $\alpha < \beta$), and then, by [D95, Proposition 45.2], there is a special $0 < s \in G$ with value V_λ, such that $s \wedge (g - s) = 0$. The reader will note that $s \in A$, while $g - s \in B$, proving that $G = A + B$.

Using the cofinality of Δ in Λ, and all the while imitating the proof of Theorem 5.5, the reader will easily be able to verify that no disjointification of A and B is possible. We leave this as an exercise. ∎

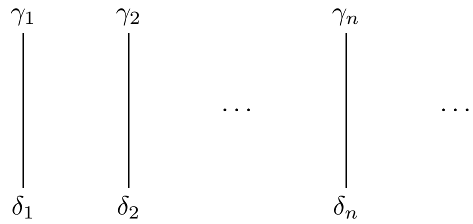
Next, what if $\mathrm{SpVal}(G)$ has more than one component? In order to generalize Theorem 5.5 fully, it is helpful to assume that G is laterally complete, as well as special valued. Recall that an ℓ-group is *laterally complete* if every set of pairwise disjoint elements has a supremum.

The next proposition completes our generalization of Theorem 5.5. One has to add an additional hypothesis to the assumption that G is special valued, as any finite valued ℓ-group is an $sd\ell$-group, for any root system $\mathrm{Val}(G)$! Example 6.6 shows that the converse of Proposition 6.5 is false, with a measure of lateral completeness; however, it is unknown whether the converse of Theorem 5.5 itself is even true. (More on that in 6.8.) And without "enough" lateral completeness the proof of Proposition 6.5 is compromised (Example 6.7).

Proposition 6.5. *Suppose that G is special valued and laterally complete, and also a $d\ell$-group. Then every component of* $\mathrm{SpVal}(G)$ *is lexed.*

Proof. Suppos there is a component $T \subseteq \mathrm{SpVal}(G)$ which is not lexed. We use Lemma 5.3: let P be the set of elements $g \in G$ for which every special component s in the expression of $|g|$ as a disjoint supremum of special components has its value in T. The polar $P^\perp$ is the set of elements x for which every special component in the expression of $|x|$ as a disjoint supremum of special components has its value off T. The lateral completeness of G implies that $G = P + P^\perp$; i.e., that P is complemented in $\mathcal{C}(G)$, and by Lemma 5.3, P is disjointifiable. But now P is a special valued ℓ-group with $\mathrm{SpVal}(G)$ order isomorphic to T, all of which contradicts Proposition 6.4. ∎

Example 6.6. Consider the root system Δ below:



G is the ℓ-subgroup of $V = V(\Delta, \mathbb{R})$ consisting of all $f \in V$ such that $f(\gamma_n)$ is eventually zero. Note that G is special valued, and the root system of special values is a copy of Δ, and therefore every one of its components is lexed.

We argue that G is not disjointifiable. To that end, let A be the convex ℓ-subgroup of all functions from $f \in G$ for which $f(\delta_n)$ is also eventually zero, and $B = \{\, f \in G : f(\gamma_n) \equiv 0 \,\}$. It is clear that $G = A + B$. Notice that B is laterally complete.

If $C, D \in \mathcal{C}(G)$ are disjoint subgroups such that $C \subseteq A$ and $D \subseteq B$, and $C + B = G = A + D$, then we may as well assume C and D are polars, and $C^\perp = D$. However, it is also easy to see that no nontrivial polar of G is archimedean, whence $D = \{0\}$, since B is archimedean. This means that $C = G$, a contradiction.

Example 6.7. Consider the root system Γ consisting of a copy of the root system Λ of Example 5.8 plus a third antichain $\{\, \nu_1, \nu_2, \dots \}$, such that each ν_i is incomparable to each λ_j and each μ_j.

G is the ℓ-subgroup of $V(\Gamma, \mathbb{R})$ consisting of all the functions f such that $f(\lambda_n)$ is eventually zero and, restricted to the well-ordered set

$$\{\, \mu_1, \nu_1, \mu_2, \nu_2, \dots \}$$

is eventually constant. Then G is special valued, and Γ itself is a copy of $\mathrm{SpVal}(G)$; its components are the singletons $\{\nu_i\}$ and Λ. But, unlike in the proof of Proposition

6.5, if e denotes the function which is 0 at each λ_i and 1 elsewhere, then e does not split into a disjoint supremum of v, vanishing on Λ, and w, vanishing on the remaining components of Γ, because the supremum of the characteristic functions of the $\{\nu_i\}$ does not exist in G.

We conclude with a remark on a problem yet to be solved.

Remark 6.8. As already admitted, we do not know whether the converse of Theorem 5.5 is true. We do not even know it for the "simplest" (infinite) root system, the antichain of natural numbers; that is, whether the ℓ-group $G = \mathbb{R}^{\mathbb{N}}$ of all real valued sequences (with pointwise ordering) is a $d\ell$-group.

We have some partial results, regarding which pairs of convex ℓ-subgroups of G for which $G = A + B$, can be disjointified. It can be shown that if $G = A + B$, with $A, B \in \mathcal{C}(G)$, and A is *majorized*, in the sense that there exists a sequence u such that for each $f \in A$, $f < nu$, for a suitable $n \in \mathbb{N}$, then B is a polar. Moreover, B is of the form

$$\{\, g \in G \,:\, g|_F = 0, \text{ for some finite } F \subseteq \mathbb{N}\,\}.$$

In this case the pair $B^{\perp}$ and B strongly disjointifies A and B.

A more detailed discussion of this example will be taken up elsewhere.

References

[BCD86] R. N. Ball, P. Conrad & M. Darnel, *Above and below subgroups of a lattice-ordered group.* Trans. AMS **297** (No. 1) (September 1986), 1-40.

[Ba97] B. Banaschewski, *Pointfree topology and the spectrum of f-rings.* In *Ordered Algebraic Structures.* W. C. Holland & J. Martinez, Eds.; (1997) Kluwer Acad. Publ., 123-148.

[BKW77] A. Bigard, K. Keimel, & S. Wolfenstein, *Groupes et Anneaux Réticulés*, Lecture Notes in Mathematics **608**, Springer-Verlag (1977), Berlin-Heidelberg-New York.

[Bi67] G. Birkhoff, *Lattice Theory.* AMS Colloq. Publ. **XXV** (1967), Providence, RI.

[C65] P. F. Conrad, *The lattice of all convex ℓ-subgroups of a lattice-ordered group.* Czech. Math. Jour. **15 (90)** (1965), 101-123.

[D95] M. Darnel, *Theory of Lattice-Ordered Groups*, Marcel Dekker (1995), New York.

[GJ76] L. Gillman & M. Jerison, *Rings of Continuous Functions.* Grad. Texts Math. **43**, Springer-Verlag (1976), Berlin-Heidelberg-New York.

[HS79] H. Herrlich & G. Strecker, *Category Theory.* Sigma Series Pure Math. **1** (1979), Heldermann Verlag, Berlin.

[J82] P. J. Johnstone, *Stone Spaces.* Cambridge Studies in Adv. Math, **3** (1982), Cambridge Univ. Press.

[M72] J. Martínez, *Unique factorization in partially ordered sets.* Proc. AMS **33** (No. 2), (June 1972), 213-220.

[M73] J. Martínez, *Archimedean lattices.* Alg. Univ. **3** (fasc. 2) (1973), 247-260.

[M90] J. Martínez, *The closed subgroup problem for lattice-ordered groups.* Arch. Math. **54** (1990), 212-224.

[M06] J. Martínez, *A theorem of Conrad, framed, without Choice.* Submitted.

[MZ03] J. Martínez & E. R. Zenk, *When an algebraic frame is regular.* Alg. Univ. **50** (2003), 231-257.

[MZ05] J. Martínez & E. R. Zenk, *Yosida frames.* To appear, Jour. of Pure & Appl. Alg.

[MZ06] J. Martínez & E. R. Zenk, *Regularity in algebraic frames.* Submitted.

[Mo54] A. Monteiro, *L'Arithmétique des filtres et les espaces topologiques.* Segundo Symposium de Matemática; Villavicencio (Mendoza) (1954), 129-162.

[PT01] M. C. Pedicchio & W. Tholen, *Special Topics in Order, Topology, Algebra and Sheaf Theory.* Cambridge Univ. Press (2001), Cambrdge, UK.

[ST93] J. T. Snodgrass & C. Tsinakis, *Finite-valued algebraic lattices.* Alg. Univ. **30** (1993), 311-318.

Department of Mathematics, University of Florida, P. O. Box 118105
Gainesville, FL 32611-8105, USA
jmartine@math.ufl.edu