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University of Florida ∗ Vanderbilt University

Frames: Algebra vs. Topology.
ABSTRACTS

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The Tutorials.

Richard N. Ball, University of Denver

The Briar Patch **(Joint work with Joanne Walters-Wayland)**

Let κ be a regular cardinal or the symbol ∞. A *κ-set* is a set of cardinality strictly less than κ, and a *κ-frame* is a bounded lattice in which κ-sets have joins which commute with binary meets, and κ-morphisms respect κ-joins, binary meets, and top and bottom. Thus an ω_1-frame is what is usually termed a σ-frame, and an ∞-frame is a what is usually termed a frame. A κ-frame is regular if each of its elements is the join of a κ-set of elements well below it.

Regular κ-frames are moncoreflective in regular λ-frames for $\kappa \leq \lambda$, so that a regular frame may be viewed as the union of these coreflections. Furthermore, these coreflections carry a great deal of information about the frame containing them. For example, a lot of information about a completely regular frame is carried by the smallest of these coreflections, the cozero of the frame. What is interesting and important is that membership in the cozero part can be determined by the intrinsic properties of the element itself, rather than by reference to its containment in some otherwise unspecified regular ω_1-subframe; this is Johnstone's notion of a scale. We will show that this remains true for any κ. An interesting notion here is that of a κ-briar, a particularly prickly poset, and of the κ-briar patch, the free frame generated by the κ-briar. This serves as the free regular κ-frame over a single generator, and is the κ-counterpart of a scale. Aside from the obvious cardinal generalizations of the theory of completely regular frames and their cozero parts, there arises from these ideas a number of useful invariants for the study of frames. One of the many apparent consequences of this generalization is the κ-Lindelöf coreflection of a frame, a construct which sheds light on the structure of the assembly NL of nuclei on a frame L.

Bernhard Banaschewski, McMaster University, Canada

Essential completion

The notion of an essential extension will be considered in the full subcategories **CRFrm, C,** and **K** of the category **Frm** of all frames, given, respectively, by: the complete regular frames, the completely regular frames complete in their real uniformity, and the compact completely regular frames. Specifically, the essentially complete objects will be characterized and the existence of unique (up to isomorphism) essential completions will be established for each of these categories.

It should be noted that, in the case of **K**, what is shown here is a familiar fact, essentially due to Gleason (1958), with the added feature that essentially complete = injective. The other two cases are new and, by way of contrast, there one has only trivial injectives. These results will then be applied to the category **W** of archimedean ℓ-groups with weak order unit by means of the function ring functor $\mathfrak{R}$ which induces an equivalence between **C** and the subcategory of **W** given by the $\mathfrak{R}L$. As a consequence, this provides unique essential completions in **W**, originally shown to exist by Conrad (1971), naturally using rather different methods.

In addition, analogous applications are made to the following subcategories of **W**: $\mathbf{W}^*$, given by all *bounded* $A \in \mathbf{W}$ (any $|a| \leq n \cdot 1$, for some n); $\mathbf{W}_S$, given by all $A \in \mathbf{W}$ with *singular* unit ($a \wedge (1-a) \leq 0$, for all a); and $\mathbf{W}_S^* = \mathbf{W}_S \cap \mathbf{W}^*$, where the singular cases work with $\mathfrak{Z}L$, determined by the integer-valued functions on L, instead of $\mathfrak{R}L$.

Warren Wm. McGovern, Bowling Green State University

A survey of algebraic frames

With the study of algebraic frames hitting a new high point in the literature we aim to survey the recent results with a specific eye towards the applications of the theory to other algebraic arenas. In particular, we hope to motivate the discussion through the use of several well-studied examples: the frame of radical ideals of commutative ring with identity, the frame frame of convex ℓ-subgroups of a lattice-ordered group, the frames of multiplicative filters and noetherian filters of a commutative ring with identity, the frame of z-ideals of $C(X)$, etc.

Aleš Pultr, Charles University, Czech Republic

Uniformity and approximation

1. Sublocales.

Various representations (surjective homomorphisms, congruences, nuclei, in particular sublocale sets). Advantages of sublocale sets (true sub-locales; structure of the co-frame; closure, density theorem; etc.). Sublocales vs. subspaces in the spatial case. Complemented sublocales.

2. Enrichment of the localic structure.

Uniformity and nearness, discussion. Uniform sublocales. Completion. Advantages of the point-free approach. Cauchy filters and Cauchy points. Metrizability and associated facts.

3. Approximation and special filters.

Approximate maps. Perhaps a few remarks on frame (localic) approach vs. the domain one (domains and Scott information systems). Localic morphisms as approximate maps. Approximate maps in the uniform resp. nearness context. General remarks on filters apt to represent something like a point (prime and completely prime filters, shaving filters, Cauchy filters), The associated spectra and similar constructions.

Distinguished Contributions.

Frederick K. Dashiell, Jr., Los Angeles, CA

The equivalence of Boolean algebras carrying a subordination relation and perfect irreducible surjections

We describe a pointfree approach to covers in topology. A *cover* of a space X is a completely regular space Y together with a perfect irreducible surjection (*covering map*) $\pi : Y \twoheadrightarrow X$. For an arbitrary c.r. X (and sometimes just regular X), there exist covers which are minimal among certain special classes of spaces such as extremally disconnected, basically disconnected, and quasi-F spaces. In some cases, the mininal covers in these classes can be characterized as maximal covers when the covering map is required to have certain additional properties. In an attempt to address some of the open questions in this area, we describe Boolean algebras equipped with an auxiliary binary relation used by H. deVries [1] to extend Stone duality to arbitrary compact spaces, and use this to describe the characterization of covers given by V. V. Fedorchuck [2].

A *subordination* on a Boolean algebra B is a binary relation $\ll$ on B which satisfies the following 5 axioms:

S1 $a \ll b \Longrightarrow a \leq b$

S2 $a \ll b \Longrightarrow b' \ll a'$

S3 $a \ll b \Longrightarrow \exists\, c$ with $a \ll c$ and $c \ll b$

S4 $a \neq 0 \Longrightarrow \exists\, b \neq 0$ with $b \ll a$

S5 $a_1 \ll b$ and $a_2 \ll b \Longleftrightarrow a_1 \vee a_2 \ll b$

If B is a complete Boolean algebra with a subordination, deVries constructs a unique compact Hausdorff space Y such that B is isomorphic to the regular open algebra $R(Y)$ of Y and $a \ll b$ if and only if a is completely separated from b' as regular open subsets of Y. If you start with a c.r. space X, and consider a *topological* subordination $\ll$ on $R(X)$ in which axiom S4 is enhanced to recognize the space as follows:

S4top $x \in a \Longrightarrow \exists\, b$ with $x \in b \ll a$,

the resulting compact deVries space contains a dense subspace Y which is a perfect irreducible preimage of X, i.e., Y is a cover of X. Conversely, every cover $\pi : Y \twoheadrightarrow X$ induces a Boolean isomorphism between $R(Y)$ and $R(X)$, and every compactification of Y induces a natural subordination on $R(Y)$, defined as above by complete separation, which the covering map π carries to a topological subordination on $R(X)$. Thus the study of covers becomes the study of topological subordination relations on the algebras

$R(X)$. In the pointfree setting, this translates to the study of *frame* subordination relations on the Booleanization $B(L)$ of a frame L, where $B(L) = \{a^* : a \in L\}$, and the pointfree version of S4top becomes:

S4pf $a = \vee_L\{b : b \ll a\}$ for all $a \in B(L)$.

References (partial list)

[1] H. de Vries, Compact Spaces and Compactifications, Van Gorcum & Comp. N.V., Assen, 1962

[2] V. V. Fedorchuk, Perfect Irreducible Mappings and Generalized Proximities, Mat. Sbornik 76 (1968), 489-508.

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Anthony W. Hager, Wesleyan Uinversity

A frame-theoretic feature of the ℓ-group of Baire functions

ker is the functor from the category **W**, of archimedean ℓ-groups with weak unit, onto Lindelöf completely regular frames ($\ker G$ is the frame of kernels of **W**-maps out of G). For an embedding e in **W**,when is $\ker(e)$ an isomorphism, surjection, injection,etc.? Madden, et. al., have analyzed the situation of isomorphism,and Ball-Hager that of surjection. In each of these cases there is a well-understood maximum such e. However, saying $\ker(e)$ is injective puts no bound on the size of the codomain $\mathrm{cod}(e)$, so we restrict (somewhat naturally) to epic e, and $\mathrm{cod}(e)$ having no further epic extension: e is an "epicompletion" (of its domain).

Let X be a Tychonoff space, $C(X)$ as usual, and b the embedding of $C(X)$ into $B(X)$, the Baire functions on X.

THEOREM. An epicompletion e of $C(X)$ has $\ker(e)$ injective iff $h \cdot e = b$ for some (unique) h.

I don't know if an arbitrary object of **W** has a similarly distinguished extension.

Peter Jipsen, Chapman University

On domain semigroups, stably supported quantales and frames

Domain semigroups are semigroups with an additional unary operation d that satisfies the identities $d(x)x = x$, $d(xd(y)) = d(xy)$, $d(d(x)y) = d(x)d(y)$ and $d(x)d(y) = d(y)d(x)$. They occur as reducts of relation algebras and quantales where $d(x) = e \wedge x\top$, and in inverse semigroups where $d(x) = xx^{-1}$. Moveover they capture modal operators algebraically via $\langle x \rangle p = d(xp)$ and hence can express Hoare triples $\{p\}x\{q\}$ by $p = pd(xq)$. A *stably supported quantale* ([4]) is a unital involutive quantale expanded with an operation d that satisfies the above identities. In a stably supported quantale the elements below the unit e form a frame with $x \wedge y = xy$, and every frame arises in this way. The complemented elements in this frame are Boolean predicates that can be used for modeling the programming constructs "if p then x else y" by $px \vee \bar{p}y$, and "while p do x" by $(\bigvee_{n=0}^{\omega}(px)^n)\bar{p}$, where $z^0 = e$ and $z^{n+1} = zz^n$. Hence in this model of program semantics the frame contains predicates over a state space and the quantale contains programs that transform predicates by multiplication.

We give a concrete description of free domain semigroups and how they extend to domain semirings and stably supported quantales. The subvariety of twisted domain semigroups (also called guarded semigroups) consists precisely of (algebras isomorphic to) semigroups of partial functions on some set X with $d(f) = \{(x, x) : x \in \mathrm{dom}(f)\}$, and the free objects have been described by Manes [3]. Free domain semigroups can be represented as semigroups of binary relations and hence are embedded in distributive stably supported quantales. However the class of domain semigroups that have relational representations is a proper subquasivariety and was recently shown to be nonfinitely axiomatizable by Hirsch and Mikulas [2].

References

[1] J. Desharnais, P. Jipsen and G. Struth, *Domain and antidomain semigroups*, in "Relations and Kleene Algebra in Computer Science", R. Berghammer et al. (Eds.), Lecture Notes in Computer Science, Vol. 5827, Springer-Verlag (2009), 73–87.

[2] R. Hirsch and S. Mikulas, *Axiomatizability of representable domain algebras*, submitted, 2009.

[3] E. Manes, *Guarded and banded semigroups*, Semigroup Forum, Vol. 72, (2006), 94–120.

[4] P. Resende, *Étale groupoids and their quantales*, Advances in Mathematics, 208 (2007), 147–209.

Peter T. Johnstone, Cambridge University, U. K .

DeMorganization: a challenge from topos theory to locale theory

It is well known that every locale has a smallest dense sublocale, its Booleanization. The Booleanization is not the largest Boolean sublocale (no such thing exists), but it can be described as the largest dense sublocale which is Boolean. Recently, it has emerged from work of my student Olivia Caramello in topos theory that every locale has a largest dense sublocale satisfying De Morgan's law, which obviously deserves to be called its DeMorganization. However, at present the only proof we have of the existence of this sublocale is topos-theoretic.

In this talk I shall present some examples of DeMorganizations, and discuss the obstacles to finding a direct locale-theoretic proof of its existence.

James Madden, Louisiana State University

Presentations of frames by generators and relations: old and new applications

Johnstone's book, *Stone Spaces*, describes locales that arise in several representation theorems in terms of generators and relations. This inspired the treatment of the localic Yosida theorem that I presented in the early 1990s. In all these examples, the functor from algebras to frames factors as a functor to lattices (possibly with some additional structure) followed by the ideal functor from lattices to frames. I described other examples of frames "free over a lattice" in my work on kappa-frames. In this talk, I will review the main themes in this work, and point out some new applications. Among other things, I will describe how Eudoxus' theory of proportion can be employed to produce a representation theorem for archimedean ℓ-groups that unifies other proposed representations.

Jorge Martínez, University of Florida

Epicompletion in frames with skeletal maps **(Joint work with Eric R. Zenk)**

A frame homomorphism h is *skeletal* if $x^{\perp\perp} = 1$ implies that $h(x)^{\perp\perp} = 1$. We consider the notion of functorial epicompletion in a number of categories of compact frames with skeletal frame maps. It is not hard to show that the absolute of a compact regular frame is the epicompletion in $\mathfrak{KRegS}$, the category of compact regular frames with skeletal maps in the subcategory $\mathfrak{SPRegS}$ whose objects are strongly projectable.

The goal is to extend this epicompletion to a monoreflection of categories whose objects are, in an algebraic sense, archimedean, with skeletal maps. Such a monoreflection is constructed via a pushout. What is not clear is when – if ever – it is the maximum monoreflection.

There are reasonable extensions of the absolute to such categories for which the construction does *not* yield the least epireflective class. But there are no examples of smaller monoreflective ones either.

Christopher J. Mulvey, University of Sussex, U. K.

Algebraic and topological aspects of étale groupoids

In a recent paper in *Advances in Mathematics*, Pedro Resende introduced an intriguing, and unexpected, connection between involutive quantales and groupoids, to obtain a characterisation of étale groupoids, that is, of groupoids

$$G_1 \rightrightarrows G_0$$

in the category of locales for which the domain (or equivalently, the codomain) map is a local homeomorphism. In turn, this result is applied to obtain a number of other interesting characterisations of these groupoids. In this talk, I will revisit his proof to examine the properties of the quantales involved that lead to the characterisation obtained, showing in particular that introducing the concept of a partially Gelfand quantale allows the characterisation to be simplified considerably.

Jorge Picado, University of Coimbra, Portugal
Rings of (extended) real functions in frames

In a recent paper with J. Gutiérrez García and T. Kubiak [Localic real functions: a general setting, *Journal of Pure and Applied Algebra* 213 (2009) 1064-1074] we introduced a point-free framework for continuous, semicontinuous and even more general real-valued functions that has proved to be the right setting for formulating extensions of the well-known classical insertion theorems.

In this talk we will present an overview of the recent developments of the theory and the new perspectives this development opens up in dealing with extended (continuous) real functions, focusing mainly on work in progress with Gutiérrez García on the problem of how to deal with algebraic operations. We acknowledge fruitful conversations with Bernhard Banaschewski on the topic.