On α-Regular Archimedean f-Rings[1]

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For Charles Holland, on his 70th or his 75th birthday,
whichever he prefers!

Abstract. For a commutative ring A with identity, and for infinite cardinals α as well as the symbol ∞, which indicates the situation in which there are no cardinal restrictions, one defines A to be *α-regular* if for each subset D of A, with $|D| < \alpha$ and $de = 0$, for any two distinct $d, e \in D$, there is an $s \in A$ such that $d^2 s = d$, for each $d \in D$, and if $xd = 0$, for each $d \in D$, then $xs = 0$,

This paper studies α-regular archimedean f-rings, relative to lateral α-completeness. The main result is that the operator $l(\alpha)$ that gives the lateral α-completion commutes with b, the reflection that closes an f-ring with respect to bounded inversion. An f-ring is α-regular if and only if it has bounded inversion and is laterally α-complete, and the operator that creates the α-regular hull is $r(\alpha) = b \cdot l(\alpha)$.

It is shown that the space $\mathrm{m}r(\alpha)A$ of all maximal ℓ-ideals of $r(\alpha)A$ is the same as that of the α-projectable hull. Finally, $r(\alpha)A$ contains the ring of α-quotients, and necessary conditions are given for them to coincide.

Introduction.

The concept of strong ω_1-regularity, which played an important role in [HM02a], is generalized for arbitrary regular, uncountable cardinals. In this paper, "ring" means "commutative ring with identity". The objective is to study the relationship between this new concept and lateral α-completeness and bounded inversion in archimedean f-rings. Prominent in the discussion will be the functorial extension b, which embeds an f-ring in one satisfying the bounded inversion property.

The ambient category in this article is **Arf**, the category of archimedean f-rings with identity and ℓ-homomorphisms which preserve the identity. The subclasses of **Arf** that will be discussed are hull classes, and we shall consider the application of a number of associated hull operators, in particular, the $l(\alpha)$ that form the lateral α-completion. One of the main results is that $l(\alpha)$ commutes with b, and that their composite is

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the α-regular hull. The role of the ring of α-quotients, introduced in [HM02b], is also studied.

All topological spaces are Tychonoff. The letter α will denote a regular, infinite cardinal, or else the symbol ∞, which should be taken as placing no conditions or bounds on cardinality.

We begin with a compendium of basic terms and concepts that are needed in the presentation ahead.

1 Preliminaries.

1.1 Background in Rings. Throughout this subsection A denotes a ring. The assumption of an ordering, making A an f-ring will be added later on. Frequently, the rings under discussion will be semiprime; that is, there will be no nonzero nilpotent elements.

Definition & Remarks 1.1.1. For any subset D of A, we call

- D *pairwise disjoint* if $de = 0$, for each pair of distinct $d, e \in D$, and denote
- $D^{\perp} = \{\, x \in A \,:\, dx = 0, \text{ for each } x \in D \,\}$.
- D is *dense in* A if $D^{\perp} = \{0\}$.

(a) A is *(von Neumann) regular* if for each $a \in A$ there is an $x \in A$ such that $a^2 x = a$.

(b) Suppose $\omega \leq \alpha \leq \infty$. A is *α-regular* if for each pairwise disjoint subset D of A, with $|D| < \alpha$, there is an $s \in A$ such that $d^2 s = d$, for each $d \in D$, and $sD^{\perp} = \{0\}$. As we will show, in semiprime rings the element s which witnesses α-regularity is unique (Proposition **1.1**.3).

For $\alpha = \omega_1$ this concept is discussed in [HM02a], albeit further qualified with the extra adverb "strongly".

Elsewhere in the literature a different definition of α-regularity does not demand that $sD^{\perp} = \{0\}$. It is, in fact, not equivalent to our definition.

(c) From [Wi89, 2.2], we take the following related notion. Call A *fully regular* if for each pair of disjoint subsets S and T of A such that $S \cup T$ is pairwise disjoint, there is an $x \in A$ for which $s^2 x = s$, for each $s \in S$, and $xT = \{0\}$.

(d) With S and T as in (c), we observe that $T \subseteq S^{\perp}$, and it then easily follows that any ∞-regular ring is also fully regular. Conversely, suppose A is semiprime, and $D \subseteq A$ is pairwise disjoint. Then (as A is semiprime) $D \cap D^{\#} = \emptyset$, where $D^{\#}$ denotes $D^{\perp} \setminus \{0\}$. Then, using Zorn's Lemma, select a maximal pairwise disjoint set X in $D^{\#}$, and observe that $bX = \{0\}$ if and only if $bD^{\#} = \{0\}$. Any $s \in A$ which witnesses the full regularity for the pair $\{D, X\}$, witnesses ∞-regularity for D.

We also caution that ∞-regularity is not quite the strong regularity discussed in [Wi89].

It should be obvious that if $\alpha < \beta$, then each β-regular ring is α-regular.

The following proposition gives an account of standard facts regarding regular rings. The last item is new, but also clear from the one that precedes it. The proof is omitted.

Proposition 1.1.2. *Suppose that $a \in A$ and $b \in A$, such that $a^2 b = a$. Let $e_a = ab$ and $a^* = b^2 a$. Then*

(a) *e_a is idempotent, and $Aa = Ae_a$. Furthermore, e_a is unique idempotent f with respect to (i) $Af = Aa$ and (ii) being of the form $f = ax$, with $a^2 x = a$.*

(b) *Regarding a^*, the following are equivalent for $y \in A$:*

  (i) *$y = a^*$.*
  (ii) *$a^2 y = a$ and $y = y^2 a$.*
  (iii) *$a^2 y = a$ and $ya^\perp = \{0\}$.*

(c) *Now assume that A is regular. Then every finitely generated ideal is generated by an idempotent.*

(d) *A is regular if and only if A is ω-regular.*

Here is the result concerning the uniqueness mentioned earlier.

Proposition 1.1.3. *Suppose that A is semiprime, and $D \subseteq A$ is pairwise disjoint. There is at most one $s \in A$ such that $a^2 s = a$, for all $a \in D$, and $sD^\perp = \{0\}$.*

Proof. It is easy to see that, owing to the semiprimeness of A,

$$(D \cup D^\perp)^\perp = \{0\}.$$

Now, if for $i = 1, 2$, $a^2 s_i = a$, for all $a \in D$, and $s_i D^\perp = \{0\}$, and $x = s_2 - s_1$, then, clearly, $xD^\perp = \{0\}$. Moreover, if $a \in D$, then

$$ax = a(s_2 - s_1) = e_a - e_a = 0,$$

by Proposition **1.1**.2(a). ■

Next, a brief review of (Utumi) rings of quotients. The reader is referred to [U56] and [L76] for further background on this topic.

Definition & Remarks 1.1.4. Suppose that A is a subring of B. B is a *ring of quotients of A* if for each pair $b_1, b_2 \in B$, with $b_2 \neq 0$, there is an $a \in A$ such that $ab_1 \in A$ and $a_2 b \neq 0$. Equivalently, B is a ring of quotients of A when, for each $f \in B$, the ideal of A

$$f^{-1}A \equiv \{\, a \in A \,:\, fa \in A \,\}$$

is dense in B.

Two special rings of quotients deserve mention. First, let $\mathfrak{r}(A)$ denote the set of non-divisors of zero, and

$$qA = \{\, a/d \,:\, a \in A,\ d \in \mathfrak{r}(A) \,\},$$

with the usual notions of equality and arithmetic of fractions. This is the *classical ring of quotients* of A.

The *maximum ring of quotients* QA is described as follows. For each dense ideal I of A, let $\mathrm{Hom}_A(I, A)$ stand for the A-module homomorphisms of I into A. Let $\mathfrak{D}$ denote the filter of dense ideals of A and form

$$QA = \varinjlim \mathrm{Hom}_A(I, A), \quad I \in \mathfrak{D},$$

The bonding maps in the above direct limit are understood to be domain restrictions. Then QA is a ring of quotients of A, and in fact, it is the largest ring of quotients of A.

1.2 Background on ℓ-Groups. We assume that the reader has a basic knowledge of lattice-ordered groups (abbr. ℓ-groups). Our standard references for ℓ-groups are [BKW77] and [D95]. Here all ℓ-groups will be archimedean, because the focus is on archimedean f-rings. This is also the place to remind the reader that an f-ring is a lattice-ordered ring in which $a \wedge b = 0$ and $c \geq 0$ imply that $ca \wedge b = 0$.

It is well known that an archimedean f-ring is necessarily commutative and semiprime ([BKW77, 12.3.2 & 12.3.9]).

Since the matter is key in the discussion of hull classes below, we will take the time to examine essential closure in ℓ-groups. The study of the essential closure in this context begins with Conrad, in [C71].

Definition & Remarks 1.2.1. Throughout these remarks G denotes an ℓ-group. For any subset S of G, $S^{\perp}$ stands for the *polar* of S; that is,

$$S^{\perp} \equiv \{\, a \in G \,:\, |a| \wedge |s| = 0, \forall\, s \in S \,\}.$$

The reader should not worry about our use of the same notation for polars and annihilators, as in **1.1**. In our setting of archimedean f-rings, annihilators and polars coincide,

The set $\mathcal{P}(G)$ of all polars of G is a complete boolean algebra under inclusion ([D95, 13.7]).

(a) Suppose that G is an ℓ-subgroup of the ℓ-group H. We say that G is *essential* in H, or that H is an *essential* extension of G, if for each $0 < h \in H$ there is a $0 < g \in G$ such that $nh \geq g$, for a suitable positive integer n.

The following are equivalent; this appears in [BKW77] as Theorem 11.1.15. It is the implication "(3) $\Longrightarrow$ (1)" that requires the archimedean feature. Assume that G is an ℓ-subgroup of H:

1. H is an essential extension of G.
2. If P is a nonzero polar of H, then $P \cap G \neq \{0\}$.
3. The map $P \mapsto P \cap G$ is a boolean isomorphism.

(b) G is *essentially closed* if it has no proper essential extensions. In order that there be any possibility of existence of essentially closed objects, one has to restrict the discourse to archimedean ℓ-groups. Otherwise, one can lexicographically extend to obtain a proper essential extension.

If G is an essential ℓ-subgroup of H, and H is essentially closed, H is an *essential closure* of G. In [C71], Conrad established the existence and uniqueness of essential closures. Briefly, the essential closure of G is obtained by first forming X, the Stone dual of $\mathcal{P}(G)$, and then taking $D(X)$, the group of all continuous functions with values in the extended reals $\mathbb{R} \cup \{\pm\infty\}$ (with the usual topology), which are real-valued on a dense subset U of X.

We note at this time that, by virtue of being a Stone dual of a complete boolean algebra, X is "sufficiently" disconnected to make $D(X)$ a semiprime ring with respect to pointwise operations. Indeed, $D(X)$ is an archimedean f-ring with pointwise ordering.

Here the essential closure of G will be denoted eG. Thus, if H is any essential extension of G, it is intermediate to G and eG, and we shall designate that by writing $G \subseteq H \subseteq eG$.

Since this suffices, for now, for our discussion of essential closures, we refer the reader to [C71] or [BKW77, 13.4]

1.3 Hull Classes and Operators.

Informally, a hull is a minimum essential extension with a collection of desired properties. Let us now be more precise.

As before, all ℓ-groups under consideration are assumed to be archimedean.

Definition & Remarks 1.3.1. (a) Consider a class $\mathbf{H}$ of ℓ-groups, closed under formation of ℓ-isomorphic copies. An $\mathbf{H}$-*hull* is a function assigning to each ℓ-group G an extension hG, such that

(i) $G \subseteq hG$ is an essential extension, with $hG \in \mathbf{H}$, and

(ii) $G \subseteq H \subseteq eG$, and $H \in \mathbf{H}$, together imply that there exists an ℓ-embedding $u : hG \longrightarrow H$, extending the identity on G.

If there is such an h associated with the class $\mathbf{H}$, we call the latter a *hull class* with *hull operator* h. One also uses the phrase "each ℓ-group G has an $\mathbf{H}$-hull" when such a hull operator exists.

As in Proposition 2.4 of [HM99b] (for the category $\mathbf{W}$ of ℓ-groups with a designated weak unit), one shows that each ℓ-group G has an $\mathbf{H}$-hull if and only if $\mathbf{H}$ is *essentially intersective*; that is, for each essentially closed ℓ-group E, and each collection $\mathfrak{A}$ of subobjects of E, such that $\mathfrak{A} \subseteq \mathbf{H}$ and $\bigcap \mathfrak{A}$ is essential in E, then $\bigcap \mathfrak{A} \in \mathbf{H}$. Note that any hull class contains all the essentially closed ℓ-groups. In particular, every hull class is nontrivial.

To conclude these observations, note that if $\mathbf{H}$ is a hull class, then for each G,

$$hG = \bigcap \Big\{ H \,:\, G \subseteq H \subseteq eG,\ H \in \mathbf{H} \Big\}.$$

As has already been observed, essentially closed ℓ-groups are, in fact, f-rings. Moreover, if A is an **Arf**-object, then the embedding $A \subseteq eA$ is as a subring. Thus, what we have just described for ℓ-groups, can be formulated *mutatis mutandis* for the category **Arf** of archimedean f-rings with identity.

The principal result of this paper asserts that two hull operators commute and that their composite is the hull operator associated with the intersection of the two given classes. This a general phenomenon, not having much to do with ℓ-groups. We address that now in the proposition that follows. The content of the proposition is well known, and the proof is omitted.

Proposition 1.3.2. *Suppose that $\mathbf{K}$ and $\mathbf{L}$ are hull classes with their respective hull operators k and l. Then*

(a) $\mathbf{K}\cap\mathbf{L}$ *is a hull class; the hull operator associated with it is obtained by transfinitely iterating the composite $l \cdot k$.*

(b) *If $l\mathbf{K} \subseteq \mathbf{K}$, then $k \cdot l \leq l \cdot k$ – which is to say that each $k(lG) \subseteq l(kG)$.*

(c) *If $l\mathbf{K} \subseteq \mathbf{K}$ and $l\mathbf{K} \subseteq \mathbf{K}$, then $k \cdot l = l \cdot k$. This is the hull operator of the intersection $\mathbf{K} \cap \mathbf{L}$.*

2 b vs. $l(\alpha)$

Throughout this section α denotes a regular, *uncountable* cardinal, except in the few instances where the contrary is spelled out, or the symbol ∞.

2.1 Lateral α-Completion. We shall describe the hull operator $l(\alpha)$, that is, the lateral α-completion. The idea is to mimic the arguments given in [HM99a, §2] for the case $\alpha = \omega_1$.

We begin, however, with projectability, to which lateral completeness is intimately related. Throughout, A denotes an archimedean ℓ-group.

Definition & Remarks 2.1.1. (a) Recall that A is *α-projectable* if for each subset S of size $< \alpha$, $A = S^{\perp\perp} + S^{\perp}$. For ω, we have that "ω-projectable" *is* "projectable"; that is to say, $A = a^{\perp\perp} + a^{\perp}$, for each $a \in A$.

If $a \in S^{\perp\perp} + S^{\perp}$ we write $a = a[S] + a[S^{\perp}]$, where $a[S] \in S^{\perp\perp}$ and $a[S^{\perp}] \in S^{\perp}$. $a[S]$ is called the *projection of a on S*. It should be observed that if A is laterally α-complete then it is also α-projectable (3.2, [HM96]).

The class $\mathbf{P}(\alpha)$ of α-projectable ℓ-groups is a hull class, with hull operator $p(\alpha)$.

The following proposition describes the operator $l(\alpha)$ in a way that will suit our purposes and avoid transfinite induction. Implicit in the proof, however, is the fact that essentially closed objects are laterally ∞-complete.

Proposition 2.1.2. *The class $\mathbf{L}(\alpha)$ of laterally α-complete ℓ-groups is a hull class, and if $l(\alpha)A$ is the hull of A, then,*

$$l(\alpha)A^{+} = \Big\{ \bigvee_{i \in I} a_i[b_i] \,:\, 0 \le a_i \in A,\ \{\, 0 \le b_i \,:\, i \in I \,\} \subseteq eA \text{ pairwise disjoint},\ |I| < \alpha \Big\},$$

Proof. It is known that $\mathbf{L}(\alpha)$ is a hull class ([HM99b, 2.9]).

Now let P denote the set of suprema on the right side of the identity we wish to prove. By recasting the arguments in the proof of Theorem 2.3 of [HM99a] for the cardinal α, it is shown that (i) $P - P$ is an ℓ-subgroup of $l(\alpha)A$, and (ii) $(P - P)^{+} = P$.

To complete the proof, it suffices to show that $(P - P)^{+}$ is laterally α-complete. For this it is enough to show that P itself is closed under suprema of fewer than α pairwise disjoint elements. This requires α to be a regular cardinal; otherwise, one follows the arguments of [HM99a, p. 251], given there for countable disjoint sets. ∎

What we want out of the foregoing is the counterpart to Corollary 2.4 of [HM99a].

Corollary 2.1.3. *For each* **Arf**-*object A, $l(\alpha)A$ is an f-ring, such that the inclusions $A \subseteq l(\alpha)A \subseteq eA$ are as f-subrings.*

Proof. For brevity let us keep the notation $P = l(\alpha)A^{+}$ of the preceding proof. It suffices to show that $P^2 \subseteq P$, under the multiplication of eA. Because if this is the case, then $l(\alpha)A^2 = (P - P)^2 \subseteq (P - P) = l(\alpha)A$, which shows that $l(\alpha)A$ is a subring of eA. And then, since $A^{+} \subseteq P$, it is clear that A embeds in $l(\alpha)A$ as a subring.

So all that remains is showing that P is closed under the multiplication of eA. This is easily seen, by observing that

$$a_1[b_1]a_2[b_2] = a_1a_2[b_1 \wedge b_2],$$

for any positive $a_i, b_i \in eA$, and applying [BKW77, 12.3.17] to distribute the expression $\vee_{i\in I} a_i[b_i]$ over $\vee_{i\in I} c_i[d_i]$, with the a_i, b_i, c_i, d_i as required to place those two suprema in P. Their product is evidently in P. ■

Two observations about the foregoing arguments are in order.

Remarks 2.1.4. (a) The arguments in the proof of the preceding corollary, involving the distribution of the product over existing suprema, are carried out in [C73, §4] in the context of f-rings which are not necessarily archimedean nor commutative. The citation we made in the proof to [BKW77] and [C73, 4.3] harken back to a result of Henriksen and Isbell in [HI62].

(b) The definition of P in the previous two proofs

$$P = \Big\{ \bigvee_{i\in I} a_i[b_i] \ : \ 0 \le a_i \in A,\ \{\, 0 \le b_i \ : \ i \in I \,\} \subseteq eA \text{ pairwise disjoint, } |I| < \alpha \Big\},$$

makes reference to pairwise disjoint sets in eA. As is pointed out in the similar development in [HM99a], eA may be replaced by any laterally α-complete ℓ-group B in which A is essentially embedded. For the proof of Corollary **2.1**.3, B also has to be a subring of eA, an assumption which is omitted in Corollary 2.4 of [HM99a].

One is allowed to substitute such a B for eA, because if $G \subseteq H$ is any essential extension, then existing suprema in G and H agree.

2.2 The Main Theorem.

The main result is the following theorem.

Theorem 2.2.1. *In* **Arf**, *and for each regular, uncountable cardinal* α *or* ∞,

(a) *A is α-regular if and only if it has bounded inversion and is laterally α-complete;*

(b) $b \cdot l(\alpha) = l(\alpha) \cdot b$.

Thus, the class $\mathbf{R}(\alpha)$ *of α-regular f-rings is a hull class, with hull operator* $r(\alpha) = l(\alpha) \cdot b$.

Before the proof is given, some unexplained terms and concepts should be explained.

Definition & Remarks 2.2.2. Recall that an f-ring A satisfies the *bounded inversion property* if each $a \ge 1$ in A is invertible. For each f-ring A, we have the following distinguished subring of qA, the classical ring of quotients. Put

$$bA \equiv \{\, a/d \in qA \ : \ d \ge 1 \,\}.$$

It is well known and, in any case, easy to prove that the assignment $A \mapsto bA$ is a monoreflection in the subcategory of f-rings having the bounded inversion property. In particular, b is a hull operator – because the extension $A \subseteq bA$ is essential and the class of f-rings with bounded inversion is the associated hull class. We refer the reader to [HM93].

Reflections will resurface in this presentation, but only in Theorem **2.2**.3, where we reformulate one instance of the main theorem (Theorem **2.2**.1) for two commuting monoreflections. Let it then suffice, for now, to recite the reflective property of b: we have $A \subseteq bA \subseteq qA$, and for each ring ℓ-homomorphism $f : A \longrightarrow B$, B having bounded inversion, there is a unique ring ℓ-homomorphism $\hat{f} : bA \longrightarrow B$ extending f, and, in fact, $\hat{f}(a/d) = f(a)f(d)^{-1}$, whenever $d \geq 1$.

Finally, the reader will note that if an f-ring is regular, then it necessarily has bounded inversion, since each $a \geq 1$ is a non-divisor of zero, which in a regular ring is invertible.

Now, here is the proof of Theorem **2.2**.1.

Proof. (a) We mimic the arguments in [HM02a]; see Theorem 3.2 and its proof, where our proposition is established for $\alpha = \omega_1$. We sketch the argument.

Suppose first that A is α-regular. Next, suppose that $\{\, 0 \leq a_\lambda \,:\, \lambda \in \Lambda \,\}$ is a pairwise disjoint set with $|\Lambda| < \alpha$. There is an $s \in A$ such that $sa_\lambda^2 = a_\lambda$, for each $\lambda \in \Lambda$, and $sx = 0$ whenever $xa_\lambda = 0$, for each $\lambda \in \Lambda$. Without loss of generality, $s \geq 0$. Then choose $t \geq 0$ such that $ts^2 = s$. It is then routine to verify that $a \equiv t^2 s = \vee_\lambda a_\lambda$.

Conversely, suppose that A is laterally α-complete and has the bounded inversion. By [HM02a, 3.5], A is von Neumann regular. In verifying α-regularity it suffices to do it for positive elements. So suppose that $\{\, f_i \,:\, i \in I \,\}$ is a set of pairwise disjoint elements of A, with $|I| < \alpha$. Form $f = \vee_i f_i$. Next, pick $s \in A$ such that $a^2 s = a$, and put $c = s^2 a$. Then as in the proof of Theorem 3.2, [HM02a], one checks that $f_i^2 c = f_i$, for each $i \in I$, and that $xc = 0$, whenever $xf_i = 0$, for each i. This concludes the proof of (a).

(b) It will suffice, in light of Proposition **1.3**.2, to prove that the lateral α-completion of an f-ring with bounded inversion has bounded inversion, and that if A is laterally α-complete, then bA is too.

To that end, suppose that A has bounded inversion and that $1 \leq f \in l(\alpha)A$. Then

$$f = \bigvee_{i \in I} a_i[b_i] \quad 0 \leq a_i \in A,$$

with $\{\, 0 \leq b_i \,:\, i \in I \,\} \subseteq eA$ pairwise disjoint, and $|I| < \alpha$. Since $f \geq 1$, the b_i also form a maximal pairwise disjoint set. Further, without loss of generality, we may suppose that each $a_i \geq 1$. Thus, $s_i = a_i^{-1} \in A$; then $g = \vee_i s_i \in l(\alpha)A$ and, evidently, $fg = 1$.

Conversely, suppose that the **Arf**-object A is laterally α-complete. We show that bA is too. Pick $\{\, a_i/b_i \,:\, b_i \geq 1,\, i \in I,\, |I| < \alpha \,\}$ with $a_i \wedge a_j = 0$ when $i \neq j$. As

A is projectable, we have that $b_i[a_i] \in A$, and we may form $b = \vee_i b_i[a_i]$, in A. Let $e = 1 - 1[a_i : i \in I]$, which is an idempotent. Finally, let $d = b + e$; it is easy to check that $d \geq 1$. Letting $a = \vee_i a_i$ the reader may verify that

$$\frac{a}{d} = \bigvee_i \frac{a_i}{d} = \bigvee_i \frac{a_i}{b_i}.$$

This shows that bA is laterally α-complete. ■

A second look at Theorem 2.3 of [HM99a], along with the comments in **2.1**.4(b), reveals that the cone

$$\Big\{ \bigvee_{i \in I} a_i[b_i] \ : \ 0 \leq a_i \in A, \ \{ 0 \leq b_i \ : \ i \in I \} \subseteq B \text{ pairwise disjoint, } |I| < \alpha \Big\},$$

is the positive cone of the laterally ω_1-complete reflection σA of the **Arf**-object A, provided B is taken to be βA, the functorial epicompletion of A ([HM99a, Theorem 3.1]). Corollary 2.4 of [HM99a] then guarantees that σA is again an f-ring. The reader should note, however, that in this context neither βA, nor σA, nor the composite of Theorem **2.2**.3 below are generally essential extensions of A; what makes the theorem work anyway is the fact that βA is an epimorphic extension of A.

Then, *mutatis mutandis*, the arguments in the proof of Theorem **2.2**.1 can be repeated to prove the following. In a more elaborate discussion this is proved already in Theorem 3.2 and Proposition 4.2 of [HM02a], to which we refer the reader.

Theorem 2.2.3. *In* **Arf**,

$$b \cdot \sigma = \sigma \cdot b,$$

and the composite is a monoreflection of **Arf** *in the subcategory* $\mathbf{R}(\omega_1)$ *of* ω_1*-regular* f*-rings.*

2.3 Representation of α-Regular f-Rings.

The representation we have in mind is by way of the Henriksen-Johnson Representation. We begin with a review of that.

Definition & Remarks 2.3.1. (a) Let X be a compact Hausdorff space. Recall that $D(X)$ is the set of all continuous functions f on X with values in the two-point compactification of the reals, $\mathbb{R} \cup \{\pm\infty\}$, with the additional stipulation that $f^{-1}\mathbb{R}$ is dense. It is a lattice under pointwise suprema and infima, but, in general, not a group or a ring under the appropriate pointwise operations. One uses the term *ℓ-group in $D(X)$* for an ℓ-group $G \subseteq D(X)$, when in G each sum $k = g + h$ satisfies $k(x) = g(x) + h(x)$ on a dense subset of X. One uses a similar convention with the term *f-ring in $D(X)$*.

For any archimedean f-ring A, $\mathrm{m}A$ stands for the space of maximal ℓ-ideals. It is well known that $\mathrm{m}A$ is a compact Hausdorff space under the hull-kernel topology, and that $\cap\, \mathrm{m}A = \{0\}$.

(b) Here is a formulation of the Henriksen-Johnson Representation Theorem; for the rest the reader is referred to [HJ61]:

> *Suppose that A is an archimedean f-ring. Then there is a ring ℓ-isomorphism ϕ from A onto an f-ring A' in $D(\mathrm{m}A)$, carrying the identity to the constant function 1, so that A' separates the points of* $\mathrm{m}A$.

Note that a subset $S \subseteq D(X)$ is said to *separate the points of* X if for each pair of distinct points $x, y \in X$ there is an $f \in S$ such that $f(x) \neq f(y)$. It is well known that the separation of points makes the space $\mathrm{m}A$ above unique (up to homeomorphism).

Finally, in this commentary, note that if $A = A^*$, the subring of bounded elements, then the Henriksen-Johnson Representation of A is in $C(\mathrm{m}A)$, the ring of continuous real-valued functions on $\mathrm{m}A$.

Next, we recall two concepts, which are indispensable in this context.

Definition 2.3.2. Let X be a space. Recall that an open set U which is the union of fewer than α cozerosets is called an *α-cozeroset*. X is *α-disconnected* if every α-cozeroset has an open closure. Note that "ω_1-disconnected" is synonymous with "basically disconnected", while "∞-disconnected" is the same as "extremally disconnected".

Recall that if the **Arf**-object A is laterally α-complete, then $\mathrm{m}A$ is α-disconnected ([HM96, 2.4]). In fact, we think it useful to point out the following. First, if A is given its Henriksen-Johnson Representation in $D(\mathrm{m}A)$, and for $f \in D(\mathrm{m}A)$ and each point $p \in \mathrm{m}A$, there is a neighborhood U of p and an $a \in A$, such that $f = a$ over the set U, then it is said that f is *locally in A*; $\mathrm{loc}A$ denotes the set of all functions in $D(\mathrm{m}A)$, that are locally in A.

We highlight the following facts from [HM96, §2]:

1. *The extension $A \subseteq \mathrm{loc}A$ is a hull operator, with the hull class that comprises all local* **Arf**-*objects; that is, those f-rings A for which* $\mathrm{loc}A = A$.
2. ([HM96, Corollary 2.4]) *A is α-projectable if and only if it is local and $\mathrm{m}A$ is α-disconnected.*

Finally, for the interested reader we mention (without any further comment) that for each **Arf**-object A, $\mathrm{m}p(\alpha)A$ is the minimum α-disconnected cover of $\mathrm{m}A$. (See [H89] for a fairly comprehensive account of covers, and [M04] for the specific claim made here.)

Regarding the passage from A to bA, the following should be observed, along with some of the consequences. The result brings up uniform completeness, about which we will rely on the reader's intuition, or else refer to the discussion in [HM96] on that subject.

Proposition 2.3.3. *Let A be an* **Arf**-*object, and α a regular cardinal $\alpha \geq \omega_1$ or ∞.*

(a) $\mathrm{m}A = \mathrm{m}bA$.

(b) *Assume A is uniformly complete and divisible; then the following are equivalent.*

  (i) *A is α-regular.*

  (ii) *A is laterally α-complete.*

  (iii) *$A = D(X)$, for a suitable compact α-disconnected. space X.*

(c) *For each* **Arf**-*object A,*

$$\mathrm{m}l(\alpha)A = \mathrm{m}r(\alpha)A = \mathrm{m}p(\alpha)A.$$

Proof. (a) This follows from the fact that, in the Henriksen-Johnson Representation, A separates the points of $\mathrm{m}bA$.

For (b), observe that (i) easily implies (ii), and that (iii) follows from (ii), by [HM99b, Theorem 5.5(b)]. If $A = D(X)$, as specified in (iii), then using [HM96, Proposition 5.1], A is laterally α-complete; it is also easily seen to have bounded inversion, whence we have (i).

(c) This is a consequence of (a) and Theorem **2.2**.1, together with the observation that neither the divisible hull nor the hull operator for uniform completion change the space of maximal ℓ-ideals; then apply [HM99b, Theorem 5.5(b)] once more. ■

3 Comparison with $q(\alpha)$

The hull operator $r(\alpha)$ produces rings of quotients of the base ring, and is closely related to the ring of α-quotients introduced in [HM02b]. Let us give a brief account of those rings of quotients.

3.1 The Ring of α-Quotients.

A stands for a commutative ring with identity. As usual, α denotes a regular, uncountable cardinal or ∞.

Definition & Remarks 3.1.1. (a) An ideal I of A is said to be *α-generated* if there is a generating set for I having fewer than α elements. Let $\mathfrak{D}_\alpha$ denote the filter of ideals containing a dense α-generated ideal of A. We consider again, as in **1.1**.4,

$$q(\alpha)A = \varinjlim \mathrm{Hom}_A(I, A), \quad I \in \mathfrak{D}_\alpha,$$

The bonding maps in the above direct limit are understood to be domain restrictions. Then $q(\alpha)A$ is a ring of quotients of A; note that $q(\infty)A = QA$, the maximum ring of quotients of A. We refer to $q(\alpha)A$ as *the ring of α-quotients of A*.

(b) In [Wi89] Wickstead shows that for semiprime rings $QA = A$ if and only if A is ∞-regular. This happens if and only if A is selfinjective, in the sense that each

$f \in \mathrm{Hom}_A(I, A)$ (for each ideal I of A – or, equivalently, for each *dense* ideal) can be lifted to $\hat{f} \in \mathrm{Hom}_A(A, A)$, that is to say, precisely when each $f \in \mathrm{Hom}_A(I, A)$ extends to a left multiplication by a suitable $a \in A$.

Picking up on these ideas we say that A is *α-selfinjective* if $q(\alpha)A = A$; equivalently, if and only if each $f \in \mathrm{Hom}_A(I, A)$, for each dense α-generated ideal extends to a left multiplication by some $a \in A$. It was shown in [HM02b, 2.3], that $q(\alpha)A$ is the least α-selfinjective ring of quotients of A.

In [G79, p. 105] there is a definition which is similar to our definition of ω_1-selfinjective, except that the ideal I need not be dense. In subsequent literature, this is sometimes called "$\aleph_0$-self-injective". The two definitions are not equivalent. In fact, in all instances except $\alpha = \infty$, the two concepts are different. Comparing the two is a work in progress.

The following observation about the composite $b \cdot l(\alpha)$ is of independent interest. Recall that qA stands for the classical quotient ring of A.

Proposition 3.1.2. *In* **Arf**,

$$q \leq b \cdot l(\omega_1).$$

Thus, $q \leq b \cdot l(\alpha)$.

Proof. The case for $\alpha = \infty$ is known. In any case the second statement clearly follows from the first.

Let us first show for A with bounded inversion, that $qA \leq l(\omega_1)A$. Assume that $0 \leq f = c/d$, with $d > 0$ and d regular in A. Now consider $d_n = d \vee 1/n$; then d_n^{-1} exists in A. Next, let

$$a_n = d_n[(d - 1/n)^+ \wedge (1/(n-1) - d)^+], \quad y_n = d_n^{-1}[(d - 1/n)^+ \wedge (1/(n-1) - d)^+],$$

and

$$e_n = 1[(d - 1/n)^+ \wedge (1/(n-1) - d)^+].$$

Then each e_n is an idempotent, and for each $n \in \mathbb{N}$,

$$a_n y_n = e_n = d y_n.$$

Letting $a = \vee_n y_n$, one easily checks that $da = 1$, proving that $d^{-1} \in l(\omega_1)A$. It now is clear that $f = c/d \in l(\omega_1)A$.

For an arbitrary A,

$$qA = q(bA) \subseteq l(\omega_1) \cdot bA = b \cdot l(\omega_1)A,$$

proving the proposition. ■

Corollary 3.1.3. *In* **Arf**, *for each regular, uncountable α, or ∞,*

$$q \cdot l(\alpha) = b \cdot l(\alpha) = l(\alpha) \cdot b = l(\alpha) \cdot q.$$

3.2 $q(\alpha) \leq r(\alpha)$**.** Here is the connection: "α-regular" implies "α-selfinjective".

Theorem 3.2.1. *Suppose that A is an* **Arf**-*object. Then $r(\alpha)A$ is α-selfinjective; thus, $q(\alpha)A \subseteq r(\alpha)A$.*

Proof. Evidently, the second claim is immediate from the first and the remarks in **3.1**.1(b). As to the first assertion, it suffices to show that if A has the bounded inversion property and is laterally α-complete then it is α-selfinjective. By Lemma 3.5, [HM02a], these two hypotheses imply that A is regular. We appear to need that explicitly.

As is explained in [HM02b, §5] and again in [HM03, 2.3], each $h \in \mathrm{Hom}_A(I, A)$ (for any dense ideal I) may be regarded as a left multiplication by a function – denoted by $\overline{h}$ in both references – which lies in $C(\mathrm{Coz}_{\mathbb{R}}(I))$. $\mathrm{Coz}_{\mathbb{R}}(I)$ is the union of all $\mathrm{coz}(a) \cap a^{-1}\mathbb{R}$, over $a \in I$; it is a dense α-cozeroset if I is α-generated. The trick here is to show that $\overline{h}$ can be "extended" to an element of A. For simplicity we identify h as homomorphism with $\overline{h}$. It should also be clear that it suffices to prove our claim for $h \geq 0$.

One final preamble: we identify A with its image under the Henriksen-Johnson Representation.

So suppose that $h \in \mathrm{Hom}_A(I, A)$, with I α-generated. Since A is von Neumann regular, we may assume that I has a generating set of idempotents $\{\, e_\lambda \,:\, \lambda \in \Lambda \,\}$, with $|\Lambda| < \alpha$. Note that each $he_\lambda \in A$. We proceed, transfinitely, to disjointify the idempotents and, simultaneously, to demonstrate that h extends (as a homomorphism) to all of them. Assume that Λ is well ordered. Let $f_1 = e_1$. Assume for a $\nu \in \Lambda$ that we have chosen $f_1, \ldots, f_\mu, \ldots$, with $\mu < \nu$, such that the f_μ are pairwise disjoint and $hf_\mu \in A$, for each $\mu < \nu$. Put

$$f_\nu = e_\nu\Big(1 - \Big(\bigvee_{\mu<\nu} f_\mu\Big)\Big).$$

Note that the supremum above makes sense since A is laterally α-complete. Clearly f_ν is disjoint from its predecessors f_μ. Moreover,

$$hf_\nu = he_\nu - he_\nu(\vee_{\mu<\nu} f_\mu),$$

which makes it clear that $hf_\nu \in A$.

Arguing transfinitely then, we have a family of idempotents $\{\, f_\lambda \,:\, \lambda \in \Lambda \,\}$ which are pairwise disjoint and such that $hf_\lambda \in A$, for each $\lambda \in \Lambda$. Now note that $\vee_\lambda f_\lambda = 1$, owing to the density of I. Thus $h = \vee_\lambda hf_\lambda$ also belongs to A, thanks to the lateral α-completeness. ■

Theorem 3.5 of [HM02b] delivers a corollary to Theorem **3.2**.1. Recall first that a ring A is *α-complemented* if for each subset $S \subseteq A$ of size $< \alpha$, there is another T also of size $< \alpha$, such that $st = 0$, for each $s \in S$ and $t \in T$, and $S \cup T$ generates a dense ideal.

Corollary 3.2.2. *For an* **Arf**-*object* A *the following are equivalent.*

(a) $q(\alpha)A = r(\alpha)A$.

(b) $q(\alpha)A$ *is* α*-regular.*

(c) A *is* α*-complemented.*

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