

## BAZZONI'S CONJECTURE

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ABSTRACT. A conjecture posed 11 years ago by S. Bazzoni is solved by showing that a Prüfer domain with the property that every locally finitely generated ideal is finitely generated is, in fact, a Prüfer domain of finite character.

By a *Prüfer domain* we mean a commutative integral domain  $R$  for which every finitely generated ideal of  $R$  is invertible. In her paper [1] S. Bazzoni investigated the class semigroup of a Prüfer domain and aimed at classifying when the class semigroup of  $R$ , denoted  $\mathcal{S}(R)$ , is a Clifford semigroup (that is, for every element  $a \in \mathcal{S}(R)$  there is an element  $x \in \mathcal{S}(R)$  such that  $a = a^2x$ ). In her work she isolated the following property.

(\*) An ideal  $I$  of  $R$  is finitely generated if and only if every localization  $IR_M$  is principally generated for every maximal ideal  $M$  of  $R$ .

It is said that the ideal  $I$  of  $R$  is *locally principal* if  $IR_M$  is a principal ideal of  $R_M$  for every maximal ideal  $M$  of  $R$ . Bazzoni proved that if a Prüfer domain has finite character (that is, every nonzero element belongs to a finite number of maximal ideals) then  $\mathcal{S}(R)$  is a Clifford semigroup, and in turn, if  $\mathcal{S}(R)$  is a Clifford semigroup, then  $R$  satisfies (\*). In a later article, [2], she proved that for a Prüfer domain  $R$ , if  $\mathcal{S}(R)$  is a Clifford semigroup, then  $R$  has finite character. In [1] and then again in [2] she proposed the following.

*Conjecture:* A Prüfer domain satisfies property (\*) if and only if  $R$  has finite character.

We prove that the conjecture is true by showing that each of the two properties can be translated, via the group of invertible ideals, into a statement about lattice-ordered groups, and then show that for lattice-ordered groups these statements are equivalent. We denote the group of invertible ideals of a Prüfer domain  $R$  by  $Inv(R)$ . In [3] the authors showed that if the set of integral invertible ideals is partially-ordered by reverse inclusion, then this partial-order extends to a partial-order on all of  $Inv(R)$ . Moreover, this order is a lattice-order and makes  $Inv(R)$  into a lattice-ordered group (or  $\ell$ -group, for short) whose positive cone is precisely the set of integral invertible ideals. The positive cone of  $Inv(R)$  will be denoted by  $Inv(R)^+$ . Terms not defined explicitly below may be found in [1], [6], [5], and [3].

For an ideal  $I$  of  $R$  we let  $F_I = \{J \in Inv(R) : J \subseteq I\}$ . It is straightforward to check that  $F_I$  is a nonempty filter on  $Inv(R)^+$ . (Recall that a nonempty *filter* on a lattice  $(L, \wedge, \vee)$  is a nonempty proper subset  $F \subset L$  such that if  $f, g \in F$ , then so is  $f \wedge g$ , and if  $f \in F$  and  $h \in L$  with  $f \leq h$ , then  $h \in F$ .) Conversely, if  $F$  is a

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filter on  $\text{Inv}(R)^+$ , then the ideal  $I = \sum_{J \in F} J$  satisfies  $F_I = F$ . Therefore, there is a one-to-one correspondence between ideals of  $R$  and nonempty filters on  $\text{Inv}(R)^+$ . A *principal filter* is a filter of the form  $\{x \in L : x \geq a\}$  for some  $a \in L$ .

**Lemma 1.** *Suppose  $R$  is a Prüfer domain. The ideal  $I$  is finitely generated if and only if  $F_I$  is a principal filter.*

Now, suppose  $(G, +, 0)$  is an abelian lattice-ordered group and let  $H$  be a subgroup.  $H$  is called a *convex  $\ell$ -subgroup* if it is a sublattice and also has the property that if  $h_1 \leq g \leq h_2$  with  $h_1, h_2 \in H$ , then  $g \in H$ . The intersection of a collection of convex  $\ell$ -subgroups is again a convex  $\ell$ -subgroup and so one may speak of the convex  $\ell$ -subgroup generated by a given set. In particular, for  $g \in G^+$ , the principal convex  $\ell$ -subgroup generated by  $g$  is denoted by  $G(g)$  and is the set  $G(g) = \{h \in G : |h| \leq ng \text{ for some } n \in \mathbb{N}\}$ .

Now, if  $P$  is a convex  $\ell$ -subgroup of  $G$  then we say  $P$  is a *prime subgroup* if whenever  $a \wedge b = 0$ , then either  $a \in P$  or  $b \in P$ . It is known that the convex  $\ell$ -subgroup  $P$  is prime if and only if the factor  $\ell$ -group  $G/P$  is a totally-ordered  $\ell$ -group. Zorn's Lemma may be applied to show that every prime subgroup contains a minimal prime subgroup. We let the set of minimal prime subgroups be denoted by  $\text{Min}(G)$ . For an element  $g \in G$ , we let  $U(g) = \{P \in \text{Min}(G) : g \notin P\}$ .

For a Prüfer domain  $R$  there is a one-to-one order reversing correspondence between the nonzero prime ideals of  $R$  and the prime subgroups of  $\text{Inv}(R)$ . This correspondence takes a prime ideal  $P$  and produces the prime subgroup  $\overline{P}$  obtained by taking the convex  $\ell$ -subgroup of  $\text{Inv}(R)$  generated by the set  $\{J \in \text{Inv}(R) : J \not\subseteq P\}$ . In particular, this correspondence restricts to a bijection between the set of maximal ideals of  $R$  and the minimal prime subgroups of  $\text{Inv}(R)$ . Furthermore, for an invertible ideal  $I \in \text{Inv}(R)^+$ , the set of maximal ideals of  $R$  containing  $I$  corresponds to the minimal primes of  $\text{Inv}(R)^+$  not containing  $I$ , i.e.,  $U(I)$ . Therefore, we obtain the result:

**Theorem 2.** *Suppose  $R$  is a Prüfer domain.  $R$  has finite character if and only if for every  $I \in \text{Inv}(R)^+$ ,  $U(I)$  is a finite set.*

To understand the (\*) condition in terms of  $\ell$ -groups we first recall that for a localization  $R_M$  at a maximal ideal  $M$ ,  $\text{Inv}(R_M)$  is  $\ell$ -isomorphic to the totally-ordered group  $\text{Inv}(R)/\overline{M}$ . Which leads us to our first definition.

**Definition 3.** Let  $(G, +, 0)$  be an abelian  $\ell$ -group and suppose  $F$  is a filter on  $G^+$ . We say  $F$  is a *cold filter* if for every  $P \in \text{Min}(G)$  the filter  $P_F = \{g + P : g \in F\}$  on  $(G/P)^+$  has a minimum. (By a *minimum* we mean there is some  $f \in F$  such that for every  $g \in F$ ,  $f + P \leq g + P$ .) Notice that a principal filter is cold.

**Lemma 4.** *Suppose  $I$  is an ideal of the Prüfer domain  $R$ .  $I$  is locally principal if and only if  $F_I$  is a cold filter.*

We now are in position to translate the (\*) condition to  $\ell$ -groups.

**Proposition 5.** *Suppose  $R$  is a Prüfer domain.  $R$  satisfies (\*) if and only if the  $\ell$ -group  $\text{Inv}(R)$  satisfies the property that every cold filter is principal.*

**Remark 6.** Observe that in order to prove that Bazzoni's conjecture is true it suffices to prove the following analog statement:

Suppose  $G$  is an abelian  $\ell$ -group for which every cold filter of  $G$  is principal. Then for every  $g \in G^+$ ,  $U(g)$  is finite.

The work of Paul Conrad on lattice-ordered groups with a finite basis is pivotal. Conrad ([4]) investigated  $\ell$ -groups with Property (F):

(F) For each  $g \in G^+$ ,  $g$  is greater than at most a finite number of disjoint elements.

The main property to be gathered from such  $\ell$ -groups is [4, Theorem 5.2] which states that if  $G$  satisfies (F), then  $G$  has a basis. (An element  $b \in G^+$  is said to be *basic* if the set  $\{g \in G : 0 \leq g \leq b\}$  is totally-ordered. A *basis* is a maximal pairwise disjoint set of elements each of which is basic.) Moreover, when  $G = G(g)$  for some  $g \in G^+$  then  $G$  does not have an infinite set of pairwise disjoint elements, and so  $G$  has a finite basis. Therefore, by the Finite Basis Theorem [5, Theorem 46.12] this means that  $G$  has a finite number of minimal prime subgroups. We explain shortly why this is useful.

**Proposition 7.** *Let  $G$  be an abelian  $\ell$ -group. If every cold filter on  $G^+$  is principal, then  $G$  satisfies (F).*

*Proof.* Suppose every cold filter of  $G^+$  is principal. Let  $g \in G$  have the property that it lies above every element of the pairwise disjoint family, say  $U = \{u_\alpha\}_{\alpha \in A}$ . Let  $\hat{U} = \{u_{\alpha_1} \vee \cdots \vee u_{\alpha_n} : \alpha_1, \dots, \alpha_n \in A\}$  and

$$F = \{h \in G^+ : h \geq g - u \text{ for some } u \in \hat{U}\}.$$

Then  $F$  is a filter on  $G^+$  since if  $(g - u_1) \wedge (g - u_2) = g - (u_1 \vee u_2)$ . Let  $P \in \text{Min}(G)$  and consider  $P_F$ . Suppose there is some  $u_\alpha \in U$  with  $u_\alpha \notin P$ . Then since the set  $U$  is pairwise disjoint it follows that for every  $\beta \in A$ ,  $u_\beta \in P$ . Thus, for any minimal prime subgroup  $P$  at most one of the  $u_\alpha$  does not belong to  $P$ , whence for any  $u \in \hat{U}$ ,  $(g - u) + P = g + P$  or  $(g - u) + P = g - u_\alpha + P$  depending on whether every element of  $U$  belongs to  $P$  or  $u_\alpha$  is the unique element not belonging to  $P$ . This forces the set  $P_F = \{f + P : f \in F\}$  to have either  $g + P$  or  $g - u_\alpha + P$  as a minimum, whence  $F$  is a cold filter.

By hypothesis  $F$  must be a principal filter. Let  $h \in G^+$  be a generator for  $F$  and notice that  $h \leq g$ . Since the set  $\{g - u : u \in \hat{U}\}$  is a filter base for  $F$  it follows that  $h = g - u$  for some  $u \in U$ . Let  $u = u_{\alpha_1} \vee \cdots \vee u_{\alpha_n}$ . Then for any  $\beta \neq \alpha_1, \dots, \alpha_n$  we have

$$g - u = h \leq g - (u \vee u_\beta) \leq g - u = h.$$

We conclude that  $u_\beta = 0$  and so  $U$  is actually a finite set.  $\square$

**Proposition 8.** *Suppose  $G$  is an  $\ell$ -group for which every cold filter on  $G^+$  is principal and let  $g \in G^+$ . Then  $G(g)$  also has the property that every cold filter on  $G^+$  is principal.*

*Proof.* Let  $F$  be a cold filter on  $G(g)^+$  and extend  $F$  to a filter on  $G^+$  by taking  $\hat{F} = \{h \in G : h \geq f \text{ for some } f \in F\}$ . We claim that  $\hat{F}$  is a cold filter on  $G^+$ . Let  $P \in \text{Min}(G)$ . Then  $P' = P \cap G(g)$  is a minimal prime subgroup of  $G(g)$  and so the set  $P'_F$  has a minimum, say  $k + P' \in F$ . For any  $h \in \hat{F}$ ,  $h \geq f$  for some  $f \in F$  and so by the Isomorphism Theorem

$$h + P \geq f + P \geq k + P.$$

It follows that  $k + P$  is a minimum for  $P_{\hat{F}}$ , whence  $\hat{F}$  is a cold filter. By hypothesis,  $\hat{F}$  is principally generated, say by  $k \in G^+$ . Now, for any  $f \in F$  we have that  $k \leq f$  and so by convexity of  $G(g)$  it follows that  $k \in G(g)^+$  and that  $F$  is the principal filter generated by  $k$ .  $\square$

**Theorem 9.** *Suppose  $G$  is an  $\ell$ -group for which every cold filter on  $G^+$  is principal. For every  $g \in G^+$ ,  $U(g)$  is a finite set.*

*Proof.* We use the fact that for any nonzero  $g \in G^+$  the collection of minimal prime subgroups of  $G(g)$  is in one-to-one correspondence with  $U(g)$  (the map sends  $P \in U(g)$  to  $P \cap G(g)$ ). By Proposition 8 if  $G$  has said property then so does  $G(g)$ . By Proposition 7,  $G(g)$  satisfies (F). Now, since any infinite pairwise disjoint set of  $G(g)$  would yield an infinite pairwise disjoint family of elements beneath  $g$  it follows that  $G(g)$  does not have an infinite family of pairwise disjoint elements. Therefore,  $G(g)$  has a finite basis and so by the Finite Basis Theorem,  $G(g)$  has a finite number of minimal prime subgroups. We conclude that  $U(g)$  is finite.  $\square$

**Theorem 10.** *Suppose  $R$  is a Prüfer domain.  $R$  satisfies  $(*)$  if and only if  $R$  has finite character.*

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