MAS 4105 Test 4

1. (8 pts) Indicate whether the following are true or false.

i. Let $A \in M_{n\times n}(R)$; A is diagonalizable if for each eigenvalue of A, the dimension of the corresponding eigenspace is equal to the multiplicity of the eigenvalue. T F

ii. Let $A \in M_{n\times n}(R)$; A is diagonalizable if there exists n eigenvectors for A. T F

iii. Let V be a finite dimensional inner product space of dimension n and let β be a set of k vectors in V with $k < n$; the Gram-Schmidt process applied to β will yield a set of k distinct, orthogonal vectors. T F

iv. Let V be a finite-dimensional inner product space and let $x, y \in V$ with both x and y nonzero vectors; then $\langle x, y\rangle > 0$. T F

v. Let V be a vector space, T a linear operator on V, and x an element of V; if there exists a scalar λ such that $T(x) = \lambda x$, then x is an eigenvector of T. T F

vi. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V; then the characteristic polynomial of T_W divides the characteristic polynomial of the operator T. T F

vii. Let T be a linear operator on a finite-dimensional vector space V, and let W be the set of all elements x of V such that $T(x) = x$; then W is a T-invariant subspace of V. T F

viii. Let V be a finite-dimensional inner product space and let S be a set of orthogonal vectors in V; then the vectors in S are linearly independent. T F

2. (10 pts) Let $A = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{pmatrix}$. Find explicit representations for matrices Q, D, and Q^{-1} (with D diagonal) such that $D = Q^{-1}AQ$:

3. (10 pts) Prove that similar matrices have the same characteristic polynomial.

4. (10 pts) Let $T : P_2(R) \rightarrow P_2(R)$ be a linear transformation with

$$T(ax^2 + bx + c) = (2a + 2b + c)x^2 + (a + 3b + c)x + (a + 2b + 2c)$$

and let $\beta = \{1, x, x^2\}$ be a basis for $P_2(R)$. If the characteristic polynomial for $[T]_\beta$ is given by $-(\lambda - 1)^2(\lambda - 5)$, use this information to find a basis for $P_2(R)$ consisting of eigenvectors for T.

5. (10 pts) Let β be a basis for a subspace W of a finite dimensional inner product space V, and let $z \in V$. Prove that $z \in W^{\perp}$ if and only if $\langle z, v \rangle = 0$ for every $v \in \beta$.

6. (10 pts) Let $V = P_2(R)$ with inner product defined by $\langle \mathbf{f(x)}, \mathbf{g(x)} \rangle = \int_0^1 f(x)g(x)\,dx$. Let W be the subspace of V spanned by the vectors $\{6t, 3t - 6, 4t^2\}$. Use the Gram-Schmidt process to find an orthogonal basis for W.

Solutions:

1. i. F ii. F iii. F iv. F v. F vi. T vii. T viii. F

2. Since the matrix A is upper triangular, the eigenvalues are the diagonal elements and therefore the λ are given by $\{2, 4, 6\}$.

For $\lambda = 2$, we determine an eigenvector as follows:

$$A - 2I_3 = \begin{pmatrix} 0 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 4$, we determine an eigenvector as follows:

$$A - 4I_3 = \begin{pmatrix} -2 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda = 6$, we determine an eigenvector as follows:

$$A - 6I_3 = \begin{pmatrix} -4 & 4 & 4 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Hence for $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ we find $Q = [v_1 \ v_2 \ v_3] = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and calculate Q^{-1} as follows

$$\left(\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

so that $Q^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

3. Let $A,\ B \in M_{n\times n}(F)$ be similar. By definition there exists an invertible matrix $P \in M_{n\times n}(F)$ such that $A = PBP^{-1}$. Therefore

$$A - \lambda I_n = PBP^{-1} - \lambda I_n = PBP^{-1} - (\lambda I_n)(PP^{-1}) = PBP^{-1} - P(\lambda I_n)P^{-1} = P(B - \lambda I_n)P^{-1}.$$

Calculating determinants we find

$$det(A - \lambda I_n) = det(P(B - \lambda I_n)P^{-1}) = det(P)det(B - \lambda I_n)det(P^{-1}) = det(B - \lambda I_n)$$

since $det(P^{-1}) = 1/det(P)$. Hence by definition, the characteristic polynomials of A and B are the same.

4. We begin by determining the matrix $[T]_\beta$. From the definition of the transformation we find

$$T(1) = x^2 + x + 2 \;\rightarrow\; [T(1)]_\beta = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

$$T(x) = 2x^2 + 3x + 2 \;\rightarrow\; [T(x)]_\beta = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

$$T(x^2) = 2x^2 + x + 1 \;\rightarrow\; [T(x^2)]_\beta = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Thus $[T]_\beta = [\ [T(1)]_\beta\ [T(x)]_\beta\ [T(x^2)]_\beta\ ] = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$

From the given characteristic polynomial we conclude that the eigenvalues are given by $\{1, 1, 5\}$. To determine the eigenvectors for T we first derive a set of eigenvectors for $[T]_\beta$.

For $\lambda = 1$, we determine two eigenvectors as follows:

$$[T]_\beta - (1)I_3 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \;\rightarrow\; \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow v_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

For $\lambda = 5$, we determine an eigenvector as follows:

$$[T]_\beta - (5)I_3 = \begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

A set of eigenvectors for T are thus derived by utilizing the transformation ϕ_β^{-1}:

$$\phi_\beta^{-1}\left(\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right) = x - 2; \quad \phi_\beta^{-1}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right) = x^2 - 1; \quad \phi_\beta^{-1}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = x^2 + x + 1.$$

5. First suppose that $z \in W^\perp$; then by definition $\langle z, x \rangle = 0$ for each $x \in W$. Since $\beta \subseteq W$, this implies that $\langle z, v \rangle = 0$ for each $v \in \beta$.

Next suppose $\langle z, v \rangle = 0$ for each $v \in \beta$ and assume $\beta = \{v_1, v_2, \ldots, v_n\}$. Since β is a basis for W, each $w \in W$ can be expressed with respect to β in the form $w = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ with $c_i \in F$. Therefore,

$$\langle z, w \rangle = \langle z, (c_1v_1 + c_2v_2 + \cdots + c_nv_n) \rangle = \bar{c}_1 \langle z, v_1 \rangle + \bar{c}_2 \langle z, v_2 \rangle + \cdots + \bar{c}_n \langle z, v_n \rangle = \bar{c}_1(0) + \bar{c}_2(0) + \cdots + \bar{c}_n(0) = 0.$$

Hence z is orthogonal to each element in W and therefore by definition, $z \in W^\perp$.

6. We begin by designating the vectors $w_1 = 6t$, $w_2 = 3t - 6$, and $w_3 = 4t^2$. Hence

$$v_1 = w_1 = 6t$$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = (3t - 6) - \frac{\int_0^1 (3t-6)(6t)\,dt}{\int_0^1 (6t)(6t)\,dt} (6t) = (3t - 6) - \frac{-12}{12} (6t) = 9t - 6$$

$$\begin{aligned} v_3 = w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 &= (4t^2) - \frac{\int_0^1 (4t^2)(6t)\,dt}{\int_0^1 (6t)(6t)\,dt} (6t) - \frac{\int_0^1 (4t^2)(9t-6)\,dt}{\int_0^1 (9t-6)(9t-6)\,dt} (9t - 6) \\ &= (4t^2) - \frac{6}{12} (6t) - \frac{1}{9} (9t - 6) = 4t^2 - 4t + \frac{2}{3}. \end{aligned}$$

Thus, an orthogonal basis for V is given by $\{6t, 9t-6, 4t^2-4t+\frac{2}{3}\}$.