MAS 3114 Test 3

1. (12 pts) Indicate whether the following are true or false:

i.) The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & 0 \\ 2 & 5 & 3 \end{bmatrix}$ is similar to a diagonal matrix. T F

ii.) The standard basis vector of R^n are always orthogonal. T F

iii.) The orthogonal complement of the row space of a matrix is equal to the null space of the transpose of the matrix. always sometimes never

iv.) The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has an eigenvalue $\lambda = 2$ of multiplicity two. T F

v.) The characteristic polynomial of a real matrix can only have real roots. T F

vi.) The matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ has an eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. T F

vii.) The vectors $\begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ are orthonormal. T F

viii.) For the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ the eigenspace corresponding to $\lambda = 0$ has a basis given by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. T F

ix.) Given a vector space V, a subspace W of V, and a vector $\mathbf{u}$ in V, the orthogonal projection of $\mathbf{u}$ onto W can be the zero vector. T F

x.) Given a vector space V, a subspace W of V, a vector $\mathbf{u}$ in V, and $\hat{\mathbf{u}}$ the orthogonal projection of $\mathbf{u}$ onto W , then $||\mathbf{u} - \hat{\mathbf{u}}|| \leq ||\mathbf{u} - \mathbf{y}||$ where $\mathbf{y}$ is any vector in W. T F

xi.) Given a set of p vectors in R^n, the Gram-Schmidt orthogonalization process applied to these vectors produces a set containing p nonzero vectors. always sometimes never

xii.) The inner product of two nonzero vectors in R^n is greater than zero.
always sometimes never

2. (10 pts) Find the projection $\hat{\mathbf{y}}$ of the vector $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}$ onto the subspace spanned by the vectors $\mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{x_2} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. Verify your answer by showing that $(\mathbf{y} - \hat{\mathbf{y}}) \cdot \hat{\mathbf{y}} = 0$.

3. (10 pts) If possible, diagonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 6 & 2 \end{bmatrix}$; verify your results through matrix multiplication (i.e., show $A = PDP^{-1}$).

4. (10 pts) Find the eigenvalues and eigenvectors for the following matrix: $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Verify that you have correctly calculated the eigenvalues by showing that the product of the eigenvalues equals the determinant of the matrix. Verify that your eigenvectors are correct by showing $A\mathbf{x} = \lambda\mathbf{x}$ for each eigenvector.

5. (10 pts) Use the Gram Schmidt process to calculate a set of orthogonal vectors from the three vectors $\mathbf{x_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x_2} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$, and $\mathbf{x_3} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$. Verify that these vectors are orthogonal by calculating their dot products.

6. (5pts) Use the normal equation to find the least squares solution to the equation $A\mathbf{x} = \mathbf{b}$ given $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$.

Solutions:

1. i.) T ii.) T iii.) always iv.) T v.) F vi.) F vii.) F viii.) T ix.) T

x.) T xi.) sometimes xii.) sometimes

2. First we must find an orthogonal basis for the subspace spanned by the two vectors. Using the Gram-Schmidt process we write

$$\mathbf{v_1} = \mathbf{x_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and calculate

$$\mathbf{v_1} \cdot \mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2, \ \mathbf{v_1} \cdot \mathbf{x_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 1$$

so that

$$\mathbf{v_2} = \mathbf{x_2} - \frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - \left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ -3/2 \end{bmatrix}.$$

Next we calculate

$$\mathbf{y} \cdot \mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -4, \ \mathbf{y} \cdot \mathbf{v_2} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 3/2 \\ 1 \\ -3/2 \end{bmatrix} = 12, \ \mathbf{v_2} \cdot \mathbf{v_2} = \begin{bmatrix} 3/2 \\ 1 \\ -3/2 \end{bmatrix} \cdot \begin{bmatrix} 3/2 \\ 1 \\ -3/2 \end{bmatrix} = 11/2$$

and find

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} + \frac{\mathbf{y} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}} \mathbf{v_2} = \left(\frac{-4}{2}\right) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{12}{11/2}\right) \begin{bmatrix} 3/2 \\ 1 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 14/11 \\ 24/11 \\ -58/11 \end{bmatrix}.$$

A direct calculation (which the student should provide) verifies that $(\mathbf{y} - \hat{\mathbf{y}}) \cdot \hat{\mathbf{y}} = 0$.

3. We begin by finding the eigenvalues of the matrix. The characteristic equation is given by

$$(1 - \lambda)(2 - \lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$$

so that the eigenvalues are $\lambda = -2,\ 5$.

For $\lambda = -2$, we solve the equation

$$(A - \lambda I)\ \mathbf{x_1} = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \mathbf{x_1} = \mathbf{0}$$

and obtain an eigenvector

$$\mathbf{x_1} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

For $\lambda = 5$, we solve the equation

$$(A - \lambda I)\ \mathbf{x_1} = \begin{bmatrix} -4 & 2 \\ 6 & -3 \end{bmatrix} \mathbf{x_2} = \mathbf{0}$$

and obtain an eigenvector

$$\mathbf{x_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence we form the matrix

$$P = [\,\mathbf{x_1} \quad \mathbf{x_2}\,] = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

and calculate

$$P^{-1} = \begin{bmatrix} -2/7 & 1/7 \\ 3/7 & 2/7 \end{bmatrix}.$$

A direct verification (which the student should provide) gives

$$\begin{bmatrix} 1 & 2 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -2/7 & 1/7 \\ 3/7 & 2/7 \end{bmatrix}.$$

4. We begin by finding the eigenvalues of the matrix. The characteristic equation is given by

$$(2 - \lambda)^3 - 12(2 - \lambda) + 16 = -\lambda^3 + 6\lambda^2 = -\lambda^2(\lambda - 6) = 0$$

so that the eigenvalues are $\lambda = 0,\ 0,\ 6$.

For $\lambda = 0$, we solve the equation

$$(A - \lambda I)\,\mathbf{x} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

and obtain a general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

so that two linearly independent eigenvectors are given by

$$\mathbf{x_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{x_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 6$, we solve the equation

$$(A - \lambda I)\,\mathbf{x} = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

and obtain a solution (and hence an eigenvector)

$$\mathbf{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We know that since the columns of A are identical, they are linearly dependent so that the determinant of the matrix is zero which is also the product of the eigenvalues.

Direct calculation (which the student should provide) demonstrates that the eigenvalues and their associated eigenvectors satisfy the equation $A\mathbf{x} = \lambda\mathbf{x}$.

5. We begin by defining

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and calculate $\mathbf{v_1} \cdot \mathbf{v_1} = 2$, $\mathbf{v_1} \cdot \mathbf{x_2} = 2$, $\mathbf{v_1} \cdot \mathbf{x_3} = 6$.

Next we determine

$$\mathbf{v_2} = \mathbf{x_2} - \frac{\mathbf{v_1} \cdot \mathbf{x_2}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \left(\frac{2}{2}\right) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

and calculate $\mathbf{v_2} \cdot \mathbf{v_2} = 6$, $\mathbf{v_2} \cdot \mathbf{x_3} = -6$.

Finally we determine

$$\mathbf{v_3} = \mathbf{x_3} - \frac{\mathbf{v_1} \cdot \mathbf{x_3}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} - \frac{\mathbf{v_2} \cdot \mathbf{x_3}}{\mathbf{v_2} \cdot \mathbf{v_2}} \mathbf{v_2} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - \left(\frac{6}{2}\right) \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{-6}{6}\right) \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Direct calculation (which the student should provide) demonstrates the the vectors $\mathbf{v_i}$ are orthogonal.

6. To find the leasts squares solution, we solve the normal equation

$$A^T A\mathbf{x} = \mathrm{A}^{\mathrm{T}}\mathbf{b}$$

which in the present case can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

To solve

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

we form the augmented matrix

$$\begin{bmatrix} 3 & 3 & 6 \\ 3 & 5 & 0 \end{bmatrix}$$

which in reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \end{bmatrix}.$$

Hence the least squares solution is given by

$$\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$