MAS 3114 Test 1

1. (10 pts) Indicate whether the following are true or false:

i.) If A and B are matrices such that AB is defined, then $(AB)^T = A^T B^T$. T F

ii.) If A and B are matrices such that AB is the $n \times n$ identity matrix, then BA is also an identity matrix. T F

iii.) A system of simultaneous equations is consistent if the echelon form of the accompanying augmented matrix has at least one free variable. T F

iv.) An $n \times n$ invertible matrix must always contain n pivots when placed in echelon form. T F

v.) The echelon form of an augmented matrix of a system of linear, homogeneous equations must contain at least one free variable. T F

vi.) A bijective mapping must always be surjective. T F

vii.) If T is a transformation from R^n to R^m then there exists a matrix A such that $T(\mathbf{x}) = \mathrm{A}\mathbf{x}$. T F

viii.) Any set of five vectors in R^6 must be linearly independent. T F

ix.) Let A be the augmented matrix for a simultaneous linear system of four equations and seven unknowns. The echelon form of A must contain at least two pivots. T F

x.) Only $n \times n$ matrices have transposes. T F

2. (5 pts) Write the general solution to the following: $2x_1 - 3x_2 + 4x_3 - x_4 = 5$.

3. (5 pts) Is the following system of simultaneous linear equations consistent? Explain your answer.

$$4x_1 - 2x_2 + 7x_3 = -5$$

$$8x_1 - 3x_2 + 10x_3 = -3.$$

4. (6 pts) Find the value(s) of h for which the following vectors are linearly independent:

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}.$$

5. (10 pts) For the matrix A and the vector $\mathbf{b}$ given below, find the general solution to the matrix equation $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

6. (6 pts) If T is a linear transformation which maps R^3 to R^7 and is given by $T(\mathbf{x}) = \mathrm{A}\mathbf{x}$, what are the dimensions of $\mathbf{x}$, A, and $A\mathbf{x}$?

7. (10 pts) Find the inverse of the following matrix.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

8. (5 pts) Determine if the following matrix is invertible.

$$\begin{bmatrix} 6 & 0 & 7 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 5 & 3 & 1 \end{bmatrix}$$

9. (6 pts) Assuming that the following matrix is invertible, write a general formula for its inverse:

$$\begin{bmatrix} d & s \\ q & r \end{bmatrix}.$$

What condition insures that the matrix is invertible?

10. (6 pts) Given the following matrices A and B, calculate $(AB)^T$: $A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \\ 2 & 8 \end{bmatrix}$,

$B = \begin{bmatrix} 6 & -3 & 1 & 4 \\ 1 & -3 & 0 & 6 \end{bmatrix}.$

Solutions:

1. i.) F ii.) F iii.) F iv.) T v.) F vi.) T vii.) F viii.) F ix.) F x.) F

2. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

3. The augmented matrix for the system can be written in echelon form as

$$\begin{bmatrix} 4 & -2 & 7 & -5 \\ 0 & 1 & -4 & 7 \end{bmatrix}.$$

Since the final column does not contain a pivot, the system is consistent.

4. We form a matrix from the vectors and obtain

$$\begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ 3 & 8 & h \end{bmatrix}.$$

Placing this matrix in echelon form yields

$$\begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h-26 \end{bmatrix}.$$

The vectors will be linearly independent if there exists a pivot in each column of the above matrix. Hence the vectors are linearly independent if $h \neq 26$.

5. We begin by forming the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{bmatrix}$$

which when placed in reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Hence the solution is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \\ 1 \end{bmatrix}.$$

6. The vector $\mathbf{x}$ is 3×1, the matrix A is 7×3, and the vector $A\mathbf{x}$ is 7×1.

7. To find the inverse we construct the matrix

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

and place it in reduced echelon form to obtain

$$\begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

so that the inverse is given by

$$\begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}.$$

8. Placing the matrix in echelon form gives

$$\begin{bmatrix} 6 & 0 & 7 & 4 \\ 0 & 5 & 3 & 1 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

Since there exists a pivot in each column, the matrix is invertible.

9. The inverse is given by

$$\frac{1}{dr - sq}\begin{bmatrix} r & -s \\ -q & d \end{bmatrix}.$$

The condition for invertibility is $dr - sq \neq 0$.

10. First we calculate

$$AB = \begin{bmatrix} 36 & -18 & 6 & 24 \\ 6 & -18 & 0 & 36 \\ 20 & -30 & 2 & 56 \end{bmatrix}$$

so that

$$(AB)^T = \begin{bmatrix} 36 & 6 & 20 \\ -18 & -18 & 30 \\ 6 & 0 & 2 \\ 24 & 36 & 56 \end{bmatrix}.$$