1. Evaluate $\int_0^1 y(y^2+1)^5\,dy$.

2. Find the absolute max and min of $f(x) = (x^2+2x)^3$ on the interval $[-3, 3]$.

3. Evaluate the limit: $\lim_{x\to 0^+} \ x^2 \ln x$.

4. Find the shortest distance between the line $y = 3x + 2$ and the point (1,1).

5. Find the derivative of $y = 8e^{\sin\theta}$.

6. Determine y' given $xy^4 + x^2y = x + 3y$.

7. Find the equation of the tangent line to the function $f(x) = \frac{x^2-1}{x^2+1}$ at the point (0,-1).

8. Evaluate $\int_0^{\pi} \; x + 2\cos x \; dx$.

9. The area of a triangle increasing at a constant rate of 10 cm^2/sec. Find $\frac{dh}{dt}$ if $\frac{db}{dt} = 2\ cm/sec$ and $b = 3\ cm$, $h = 2\ cm$.

10. For the function $f(x) = \frac{x}{x+8}$ find: intervals upon which the function is increasing and decreasing; intervals upon which the function is concave upward and downward; local max and mins; inflection points; and vertical and horizontal asymptotes. Use this information to graph the function.

10. cont.

Solutions:

1. We begin with a substitution and choose $u = y^2 + 1$ so that $du = 2y\ dy$ or $(1/2)du = y\ dy$. Rewriting the integral we see

$\int\ y(y^2+1)^5\ dy = \int\ (y^2+1)^5(y\ dy) = \int\ u^5\ (1/2)du = (1/2)\int\ u^5\ du = (1/2)(1/6)u^6 = (1/12)u^6$

$= (1/12)(y^2+1)^6$. Therefore, $\int_0^1\ y(y^2+1)^5\ dy = (1/12)(y^2+1)^6|_0^1$

$= (1/12)((1)^2+1)^6 - (1/12)((0)^2+1)^6 = (1/12)(64) - (1/12)(1) = 63/12 = 21/4.$

2. We begin by finding the critical points of the function on the interval $[-3, 3]$. Setting the derivative equal to zero gives

$$f'(x) = 3(x^2+2x)^2(2x+2) = 6(x+1)(x^2+2x)^2 = 6(x+1)(x[x+2])^2 = 6(x+1)(x+2)^2x^2 = 0$$

so that the critical points of interest are given by $x = 0, -1, -2$. Next we evaluate the function at the critical points and both endpoints to obtain:

$$f(-3) = 27, \quad f(-2) = 0, \quad f(-1) = -1, \quad f(0) = 0, \quad f(3) = 3375.$$

Thus the absolute max is 3375 and occurs at $x = 3$ while the absolute min is -1 and occurs at $x = -1$.

3. We begin by noting that $(0)^2 \ln(0) = 0 \cdot -\infty$ which is indeterminate and not of the correct form for L' Hospital's rule. By rearranging we obtain

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \frac{-x^2}{2} = 0.$$

4. The distance between the point (1,1) and an arbitrary point (x, y) on the line is given by

$$D = \sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-1)^2 + ((3x+2)-1)^2} = \sqrt{(x-1)^2 + (3x+1)^2}$$

$$= \sqrt{(x^2-2x+1) + (9x^2+6x+1)} = \sqrt{10x^2+4x+2}.$$

Note that if we minimize the quantity D^2 we also minimize the quantity D. Hence we write

$$D^2 = g(x) = 10x^2 + 4x + 2$$

and seek the critical points for this function. Since $f'(x) = 20x + 4$, the only critical point is given by $x = -1/5$; we observe that the graph of $f(x)$ is a parabola which is concave upward so that the sole critical point must be an absolute min. For a value $x = -1/5$ the corresponding y value on the line is given by $y = 3(-1/5) + 2 = 7/5$; hence the minimal distance is given by

$$D = \sqrt{((-1/5) - 1)^2 + ((7/5) - 1)^2} = \sqrt{36/25 + 4/25} = \sqrt{8/5} \approx 1.265$$

5. $y' = \frac{d}{d\theta}\, 8e^{\sin\theta} = (8e^{\sin\theta})\frac{d}{d\theta}(\sin\theta) = (8e^{\sin\theta})(\cos\theta) = 8\cos\theta\; e^{\sin\theta}$.

6.

$$\begin{aligned}
\frac{d}{dx}(xy^4 + x^2y) &= \frac{d}{dx}(x + 3y) \\
\frac{d}{dx}(xy^4) + \frac{d}{dx}(x^2y) &= \frac{d}{dx}(x) + \frac{d}{dx}(3y) \\
(x)'(y^4) + (x)(y^4)' + (x^2)'(y) + (x^2)(y)' &= 1 + 3y' \\
(1)(y^4) + (x)(4y^3y') + (2x)(y) + (x^2)(y') &= 1 + 3y' \\
y^4 + 4xy^3y' + 2xy + x^2y' &= 1 + 3y' \\
4xy^3y' + x^2y' - 3y' &= -y^4 - 2xy + 1 \\
(4xy^3 + x^2 - 3)y' &= -y^4 - 2xy + 1 \\
y' &= (-y^4 - 2xy + 1)/(4xy^3 + x^2 - 3).
\end{aligned}$$

7. We begin by finding $f'(x)$:

$$f'(x) = \frac{d}{dx}\,\frac{x^2-1}{x^2+1} = \frac{(x^2-1)'(x^2+1) - (x^2-1)(x^2+1)'}{(x^2+1)^2} = \frac{(2x)(x^2+1) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

so that $f'(0) = 0$. Thus the slope-intercept equation of the tangent line is given by $y = f'(0)x + b = (0)x + b = b$; the value for b is determined by inserting the point (0,-1) into the equation of the line and obtaining $b = -1$; therefore the tangent line is horizontal with the equation $y = -1$.

8.

$$\int_0^\pi x + 2\cos x\; dx = \int_0^\pi x\; dx + \int_0^\pi 2\cos x\; dx = \int_0^\pi x\; dx + 2\int_0^\pi \cos x\; dx = (x^2/2)|_0^\pi + (2\sin x)|_0^\pi$$

$$= [(\pi^2/2) - (0^2/2)] + [(2\sin\pi) - (2\sin 0)] = [\pi^2/2] + [(2\cdot 0) - (2\cdot 0)] = \pi^2/2.$$

9. We know that for the triangle $A = (1/2)bh$ so that

$$\frac{dA}{dt} = \frac{d}{dt}\,(1/2)bh = (1/2)b'h + (1/2)bh'.$$

Inserting the values from the problem (and suppressing units for clarity) yields

$$10 = (1/2)(2)(2) + (1/2)(3)h'$$

$$10 = 2 + 3h'/2$$

$$8 = 3h'/2$$

$$h' = 16/3.$$

Therefore $\frac{dh}{dt} = (16/3)$ cm/sec.

10. For the function $f(x)$ we find

$$f'(x) = \frac{d}{dx}\,\frac{x}{x+8} = \frac{(x)'(x+8) - (x)(x+8)'}{(x+8)^2} = \frac{(1)(x+8) - (x)(1)}{(x+8)^2} = \frac{(x+8) - (x)}{(x+8)^2} = \frac{8}{(x+8)^2}$$

$$f''(x) = \frac{d}{dx}\,f'(x) = \frac{d}{dx}\,\frac{8}{(x+8)^2} = \frac{d}{dx}\,8(x+8)^{-2} = -16(x+8)^{-3}.$$

The point $x = -8$ is a vertical asymptote and direct analysis reveals that $\lim_{x\to -8^-} f(x) = +\infty$ and $\lim_{x\to -8^+} f(x) = -\infty$. In addition we note $\lim_{x\to -\infty} f(x) = 1 = \lim_{x\to +\infty} f(x)$, so that $y = 1$ is a horizontal asymptote.

Since $f''(x) = \frac{-16}{(x+8)^3}$ has no real zeros, there are no inflection points, however, a change in concavity can occur at the vertical asymptote $x = -8$. Using 'test points' we find that $f''(x) > 0$ for $x < -8$ and $f''(x) < 0$ for $x > -8$ so that function is concave upward on $(-\infty, -8)$ and concave downward on $(-8, +\infty)$.

Examining the first derivative $f'(x) = \frac{8}{(x+8)^2}$, we note that this function has no real zeros so that the only critical point occurs at the the vertical asymptote $x = -8$; hence, $f(x)$ has no local max or min. Using 'test points' we find that $f'(x) > 0$ for both $x < -8$ and $x > -8$ so that the function is increasing on $(-\infty, -8) \cup (-8, +\infty)$.