The following are a set of additional practice problems for Section 8.3. These are "nice" problems in the sense that the coefficients in the power series expansions can be expressed in a compact form. These problems illustrate many of the "tricks-of-the-trade " associated with power series solutions.

1. Find the series expansion about $x_0 = 0$ for the solution to the equation

$y'' + 9y = 0.$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 9 a_n x^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 9 a_n x^n = 0.$$

Explicitly writing the first two terms from the first series yields

$$(0) a_0 x^{-2} + (0) a_1 x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 9 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} 9 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + 9a_n]x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -9a_n/[(n+2)(n+1)],\ \ n \geq 0.$$

From this equation we can write

$$\begin{gathered}
n=0, \quad a_2 = -9a_0/[2 \cdot 1] = -9a_0/(2!) \\
n=1, \quad a_3 = -9a_1/[3 \cdot 2] = -9a_1/(3!) \\
n=2, \quad a_4 = -9a_2/[4 \cdot 3] = 9^2 a_0/(4!) \\
n=3, \quad a_5 = -9a_3/[5 \cdot 4] = 9^2 a_1/(5!) \\
n=4, \quad a_6 = -9a_4/[6 \cdot 5] = -9^3 a_0/(6!) \\
n=5, \quad a_7 = -9a_5/[7 \cdot 6] = -9^3 a_1/(7!)
\end{gathered}$$

and obtain the general formulas

$$a_{2k} = (-1)^k 9^k a_0/(2k)!, \qquad a_{2k+1} = (-1)^k 9^k a_1/(2k+1)!.$$

Thus we can write

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n}/(2n)! + a_1 \sum_{n=0}^{\infty} (-1)^n 9^n x^{2n+1}/(2n+1)!.$$

2. Find the series expansion about $x_0 = 0$ for the solution to the equation

$y'' - y = 0.$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponent of the first series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Explicitly writing the first two terms from the first series yields

$$(0)a_0 x^{-2} + (0)a_1 x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n] x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = a_n/[(n+2)(n+1)],\ n \geq 0.$$

From this equation we can write

$$\begin{aligned} n &= 0, \quad a_2 = a_0/[2 \cdot 1] = a_0/(2!) \\ n &= 1, \quad a_3 = a_1/[3 \cdot 2] = a_1/(3!) \\ n &= 2, \quad a_4 = a_2/[4 \cdot 3] = a_0/(4!) \\ n &= 3, \quad a_5 = a_3/[5 \cdot 4] = a_1/(5!) \\ n &= 4, \quad a_6 = a_4/[6 \cdot 5] = a_0/(6!) \end{aligned}$$

and obtain the general formulas

$$a_{2k} = a_0/(2k)!, \qquad a_{2k+1} = a_1/(2k+1)!.$$

Thus we can write

$$y = a_0 \sum_{n=0}^{\infty} x^{2n}/(2n)! + a_1 \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!.$$

3. Find the series expansion about $x_0 = 0$ for the solution to the equation

$y'' - 2y' + y = 0.$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=0}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponent of the first and second series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=-1}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Explicitly writing the first two terms from the first series and the first term from the second series yields

$$(0)a_0 x^{-2} + (0)a_1 x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + (0)a_0 x^{-1} - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n]x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = 2a_{n+1}/(n+2) - a_n/[(n+2)(n+1)],\ n \geq 0.$$

From this equation we can write

$$n = 0, \quad a_2 = 2a_1/2 - a_0/[2 \cdot 1] = a_1 - a_0/(2!)$$

$n = 1,\ \ a_3 = 2a_2/3 - a_1/[3{\cdot}2] = (2/3)(a_1 - a_0/(2!)) - a_1/(3!) = 3a_1/3! - 2a_0/(3!)$

$n = 2,\ \ a_4 = 2a_3/4 - a_2/[4{\cdot}3] = (2/4)(3a_1/3! - 2a_0/(3!)) - (1/[4{\cdot}3])(a_1 - a_0/(2!)) = 4a_1/4! - 3a_0/4!$

$n = 3,\ \ a_5 = 2a_4/5 - a_3/[5{\cdot}4] = (2/5)(4a_1/4! - 3a_0/4!) - (1/[5{\cdot}4])(3a_1/3! - 2a_0/(3!)) = 5a_1/5! - 4a_0/5!.$

and obtain the general formulas

$$a_k = ka_1/k! - (k-1)a_0/k! = a_1/(k-1)! - (k-1)a_0/k!, \quad k \geq 2.$$

Thus we can write

$$y = -a_0 \sum_{n=0}^{\infty} (n-1)x^n/n! + a_1 \sum_{n=1}^{\infty} x^n/(n-1)!.$$

4. Find the series expansion about $x_0 = 0$ for the solution to the equation

$(x^2+2)y'' + 4xy' + 2y = 0.$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$(x^2+2) \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 4x \sum_{n=0}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^2 \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2 \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 4x \sum_{n=0}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

Performing a shift so that the exponent of the second series is equal to n we obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=-2}^{\infty} 2(n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

Explicitly writing the first two terms from the second series yields

$$\sum_{n=0}^{\infty} n(n-1)a_nx^n+(0)a_0x^{-2}+(0)a_1x^{-1}+\sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty} 4na_nx^n+\sum_{n=0}^{\infty} 2a_nx^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_nx^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} 4na_nx^n + \sum_{n=0}^{\infty} 2a_nx^n = 0$$

$$\sum_{n=0}^{\infty} [(2(n+2)(n+1))a_{n+2} + (n(n-1) + 4n + 2)a_n]x^n = 0$$

$$\sum_{n=0}^{\infty} [(2(n+2)(n+1))a_{n+2} + (n+2)(n+1)a_n]x^n = 0$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -a_n/2, \ n \geq 0.$$

From this equation we can write

$$n = 0, \quad a_2 = -a_0/2$$

$$n = 1, \quad a_3 = -a_1/2$$

$$n = 2, \quad a_4 = -a_2/2 = a_0/2^2$$

$$n = 3, \quad a_5 = -a_3/2 = a_1/2^2$$

$$n = 4, \quad a_6 = -a_4/2 = -a_0/2^3$$

$$n = 5, \quad a_7 = -a_5/2 = -a_1/2^3$$

and obtain the general formulas

$$a_{2k} = (-1)^k a_0/(2^k), \qquad a_{2k+1} = (-1)^k a_1/(2^k).$$

Thus we can write

$$y = a_0 \sum_{n=0}^{\infty} [(-1)^n/2^n]x^{2n} + a_1 \sum_{n=0}^{\infty} [(-1)^n/2^n]x^{2n+1}.$$

5. Find the series expansion about $x_0 = 0$ for the solution to the equation

$(x^2 + 1)y'' + 2xy' = 0.$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and substituting into the above equation obtain

$$(x^2+1) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 2x \sum_{n=0}^{\infty} n a_n x^{n-1} = 0$$

$$x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 2x \sum_{n=0}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} 2n a_n x^{n} = 0.$$

Performing a shift so that the exponent of the second series is equal to n we obtain

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n} + \sum_{n=-2}^{\infty} (n+2)(n+1) a_{n+2} x^{n} + \sum_{n=0}^{\infty} 2n a_n x^{n} = 0.$$

Explicitly writing the first two terms from the second series yields

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n} + (0) a_0 x^{-2} + (0) a_1 x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n} + \sum_{n=0}^{\infty} 2n a_n x^{n} = 0$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n} + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^{n} + \sum_{n=0}^{\infty} 2n a_n x^{n} = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + (n(n-1) + 2n) a_n] x^{n} = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + n(n+1) a_n] x^{n} = 0$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -n a_n/(n+2), \ n \geq 0.$$

From this equation we can write

$$n=0,\quad a_2=-(0)a_0/2=0$$
$$n=1,\quad a_3=-a_1/3$$
$$n=2,\quad a_4=-2a_2/4=0$$
$$n=3,\quad a_5=-3a_3/5=a_1/5$$
$$n=4,\quad a_6=-4a_4/6=0$$
$$n=5,\quad a_7=-5a_5/7=-a_1/7$$

and obtain the general formulas

$$a_0=a_0;\quad a_{2k}=0,\ k\geq 1;\ a_{2k+1}=(-1)^k a_1/(2k+1).$$

Thus we can write

$$y=a_0+a_1\sum_{n=0}^{\infty}[(-1)^n/(2n+1)]x^{2n+1}.$$

6. Find the series expansion about $x_0=0$ for the solution to the equation

$y''-xy'-y=0.$

We proceed directly by writing

$$y=\sum_{n=0}^{\infty} a_n x^n$$
$$y'=\sum_{n=0}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

and substituting into the above equation obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}-x\sum_{n=0}^{\infty} n a_n x^{n-1}-\sum_{n=0}^{\infty} a_n x^n=0$$
$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} n a_n x^{n}-\sum_{n=0}^{\infty} a_n x^n=0.$$

Performing a shift so that the exponent of the first series is equal to n we obtain

$$\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_nx^n + - \sum_{n=0}^{\infty} a_nx^n = 0.$$

Explicitly writing the first two terms from the first series and the first term from the third series yields

$$(0)a_0x^{-2} + (0)a_1x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_nx^n + - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} na_nx^n + - \sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n = 0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = a_n/(n+2), \ n \geq 0.$$

From this equation we can write

$$\begin{aligned}
n = 0, &\quad a_2 = a_0/2 \\
n = 1, &\quad a_3 = a_1/3 \\
n = 2, &\quad a_4 = a_2/4 = a_0/[4 \cdot 2] \\
n = 3, &\quad a_5 = a_3/5 = a_1/[5 \cdot 3] \\
n = 4, &\quad a_6 = a_4/6 = a_0/[6 \cdot 4 \cdot 2] \\
n = 5, &\quad a_7 = a_5/7 = a_1/[7 \cdot 5 \cdot 3] \\
n = 6, &\quad a_8 = a_6/8 = a_0/[8 \cdot 6 \cdot 4 \cdot 2] \\
n = 7, &\quad a_9 = a_7/9 = a_1/[9 \cdot 7 \cdot 5 \cdot 3].
\end{aligned}$$

At first it would appear that there does not exist a compact form for representing the coefficients; however, with some imagination (and experience which comes from solving a great many of these problems) we can devise a compact formulation. For a_8 we can write

$$8 \cdot 6 \cdot 4 \cdot 2 = (2 \cdot 4) \cdot (2 \cdot 3) \cdot (2 \cdot 2) \cdot (2 \cdot 1) = 2^4 \cdot [4 \cdot 3 \cdot 2 \cdot 1] = 2^4 \cdot 4!$$

so that in general the product of the first k even integers can be expressed as

$$(2k)\cdot(2(k-1))\cdots 4\cdot 2 = 2^k\cdot k!.$$

For a_9 we can write

$$[9\cdot 7\cdot 5\cdot 3] = \frac{[9\cdot 7\cdot 5\cdot 3][8\cdot 6\cdot 4\cdot 2]}{[8\cdot 6\cdot 4\cdot 2]} = \frac{9!}{2^4\cdot 4!}$$

so that in general the product of the first $k+1$ odd integers can be written as

$$(2k+1)\cdot(2(k-1)+1)\cdots 5\cdot 3\cdot 1 = (2k+1)\cdot(2(k-1)+1)\cdots 5\cdot 3 = \frac{(2k+1)!}{2^k\cdot k!}.$$

Using this we can express the coefficients as

$$a_{2k} = a_0/[2^k\cdot k!] \qquad a_{2k+1} = [a_1(2^k\cdot k!)]/[(2k+1)!].$$

Thus the series solution to the differential equation may be written as

$$y = a_0\sum_{n=0}^{\infty} x^{2n}/[2^n\cdot n!] + a_1\sum_{n=0}^{\infty}[2^n\cdot n!]x^{2n+1}/[(2n+1)!].$$

7. Find the series expansion about $x_0 = 0$ for the solution to the equation

$(x^2+x+1)y'' + (4x+2)y' + 2y = 0.$

We proceed directly by writing

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

and substituting into the above equation obtain

$$(x^2+x+1)\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2}+(4x+2)\sum_{n=0}^{\infty} na_nx^{n-1}+2\sum_{n=0}^{\infty} a_nx^n=0$$

$$x^2\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2}+x\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2}$$

$$+4x\sum_{n=0}^{\infty} na_nx^{n-1}+2\sum_{n=0}^{\infty} na_nx^{n-1}+2\sum_{n=0}^{\infty} a_nx^n=0$$

$$\sum_{n=0}^{\infty} n(n-1)a_nx^n+\sum_{n=0}^{\infty} n(n-1)a_nx^{n-1}+\sum_{n=0}^{\infty} n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty} 4na_nx^n+\sum_{n=0}^{\infty} 2na_nx^{n-1}+\sum_{n=0}^{\infty} 2a_nx^n=0.$$

Performing a shift so that the exponent of the second,third, and fifth series is equal to n we obtain

$$\sum_{n=0}^{\infty} n(n-1)a_nx^n+\sum_{n=-1}^{\infty} (n+1)(n)a_{n+1}x^n+\sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty} 4na_nx^n+\sum_{n=-1}^{\infty} 2(n+1)a_{n+1}x^n$$

$$+\sum_{n=0}^{\infty} 2a_nx^n=0.$$

Explicitly writing the first two terms from the third series and the first term from the second and fifth series yields

$$\sum_{n=0}^{\infty} n(n-1)a_nx^n+(0)a_0x^{-1}+\sum_{n=0}^{\infty} (n+1)(n)a_{n+1}x^n+(0)a_0x^{-2}+(0)a_1x^{-1}+\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$+\sum_{n=0}^{\infty} 4na_nx^n+(0)a_0x^{-1}+\sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty} 2a_nx^n=0$$

$$\sum_{n=0}^{\infty} n(n-1)a_nx^n+\sum_{n=0}^{\infty} (n+1)(n)a_{n+1}x^n+\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty} 4na_nx^n+\sum_{n=0}^{\infty} 2(n+1)a_{n+1}x^n$$

$$+\sum_{n=0}^{\infty} 2a_nx^n=0$$

$$\sum_{n=0}^{\infty} \{[(n+2)(n+1)]a_{n+2}+[(n+1)(n)+2(n+1)]a_{n+1}+[n(n-1)+4n+2]a_n\}x^n=0$$

$$\sum_{n=0}^{\infty} \{[(n+2)(n+1)]a_{n+2}+[(n+1)(n+2)]a_{n+1}+[(n+1)(n+2)]a_n\}x^n=0.$$

From this last equation we obtain the recurrence equation

$$a_{n+2}=-a_{n+1}-a_n,\ n\geq 0.$$

From this equation we can write

$$\begin{aligned} n=0,&\quad a_2=-a_1-a_0\\ n=1,&\quad a_3=-a_2-a_1=-(-a_1-a_0)-a_1=a_0\\ n=2,&\quad a_4=-a_3-a_2=-(a_0)-(-a_1-a_0)=a_1\\ n=3,&\quad a_5=-a_4-a_3=-(a_1)-(a_0)=-a_1-a_0\\ n=4,&\quad a_6=-a_5-a_4=-(-a_1-a_0)-(a_1)=a_0\\ n=5,&\quad a_7=-a_6-a_5=-(a_0)-(-a_1-a_0)=a_1\\ n=6,&\quad a_8=-a_7-a_6=-(a_1)-(a_0)=-a_1-a_0 \end{aligned}$$

and obtain the general formulas

$$a_{3k}=a_0,\qquad a_{3k+1}=a_1\qquad a_{3k+2}=-a_1-a_0.$$

Thus we can write

$$y=a_0\sum_{n=0}^{\infty} x^{3n}+a_1\sum_{n=0}^{\infty} x^{3n+1}-(a_0+a_1)\sum_{n=0}^{\infty} x^{3n+2}$$

$$y=a_0(\sum_{n=0}^{\infty} x^{3n}-\sum_{n=0}^{\infty} x^{3n+2})+a_1(\sum_{n=0}^{\infty} x^{3n+1}-\sum_{n=0}^{\infty} x^{3n+2})$$

$$y=-a_0(x^2-1)\sum_{n=0}^{\infty} x^{3n}-a_1(x^2-x)\sum_{n=0}^{\infty} x^{3n}$$

$$y=-(a_0(x^2-1)+a_1(x^2-x))\sum_{n=0}^{\infty} x^{3n}.$$

8. Find the series expansion about $x_0=0$ for the solution to the equation

$(x^2+1)y''+3xy'+y=0.$

We proceed directly by writing

$$y=\sum_{n=0}^{\infty} a_n x^n$$

$$y'=\sum_{n=0}^{\infty} na_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

and substituting into the above equation obtain

$$(x^2+1)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 3x\sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$x^2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 3x\sum_{n=0}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n} + \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} 3na_n x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Performing a shift so that the exponent of the second series is equal to n we obtain

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n} + \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=0}^{\infty} 3na_n x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Explicitly writing the first two terms from the second series yields

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n} + (0)a_0x^{-2} + (0)a_1x^{-1} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=0}^{\infty} 3na_n x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n} + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=0}^{\infty} 3na_n x^{n} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [((n+2)(n+1))a_{n+2} + (n(n-1)+3n+1)a_n]x^n = 0$$

$$\sum_{n=0}^{\infty} [((n+2)(n+1))a_{n+2} + (n+1)^2a_n]x^n = 0$$

From this last equation we obtain the recurrence equation

$$a_{n+2} = -(n+1)a_n/(n+2),\ n \geq 0.$$

From this equation we can write

$$n=0,\quad a_2 = -a_0/2$$

$$n=1,\quad a_3 = -2a_1/3$$

$$n=2,\quad a_4 = -3a_2/4 = [3\cdot 1]a_0/[4\cdot 2]$$

$$n=3,\quad a_5 = -4a_3/5 = [4\cdot 2]a_1/[5\cdot 3]$$

$$n=4,\quad a_6 = -5a_4/6 = -[5\cdot 3\cdot 1]a_0/[6\cdot 4\cdot 2]$$

$$n = 5, \quad a_7 = 6a_5/7 = -[6 \cdot 4 \cdot 2]a_1/[7 \cdot 5 \cdot 3].$$
$$n = 6, \quad a_8 = -7a_6/8 = [7 \cdot 5 \cdot 3 \cdot 1]a_0/[8 \cdot 6 \cdot 4 \cdot 2]$$
$$n = 5, \quad a_9 = 8a_5/9 = [8 \cdot 6 \cdot 4 \cdot 2]a_1/[9 \cdot 7 \cdot 5 \cdot 3].$$

Let's examine a_8 and a_9.

$a_8 = [7{\cdot}5{\cdot}3{\cdot}1]a_0/[8{\cdot}6{\cdot}4{\cdot}2] = ([7{\cdot}5{\cdot}3{\cdot}1]{\cdot}[8{\cdot}6{\cdot}4{\cdot}2])a_0/([8{\cdot}6{\cdot}4{\cdot}2])^2 = 8!a_0/([8{\cdot}6{\cdot}4{\cdot}2])^2$.

Using the techniques from Problem 6 we find

$$8 \cdot 6 \cdot 4 \cdot 2 = 2^4 \cdot 4!$$

so that

$$a_8 = 8!a_0/(2^4 \cdot 4!)^2.$$

Next

$a_9 = [8{\cdot}6{\cdot}4{\cdot}2]a_1/[9{\cdot}7{\cdot}5{\cdot}3] = ([8{\cdot}6{\cdot}4{\cdot}2])^2a_1/([9{\cdot}7{\cdot}5{\cdot}3]{\cdot}[8{\cdot}6{\cdot}4{\cdot}2]) = (2^4{\cdot}4!)^2a_1/9!$.

From this we deduce the general formulas

$$a_{2k} = (-1)^k(2k)!a_0/(2^kk!)^2, \qquad a_{2k+1} = (-1)^k(2^kk!)^2a_1/(2k+1)!.$$

Thus we can write

$$y = a_0\sum_{n=0}^{\infty} (-1)^n(2n)!x^{2n}/(2^nn!)^2 + a_1\sum_{n=0}^{\infty} (-1)^n(2^nn!)^2x^{2n+1}/(2n+1)!.$$